NICHOLSON'S

NEW

Practical Builder,

AND

WORKMAN'S COMPANION,

IN

CARPENTRY, MASONRY, &c. &c.

WITH THE

THEORY AND PRACTICE

OF

The Five Orders,

As employed in Decorative

ARCHITECTURE.

/

THE NEW

PRACTICAL BUILDER,

AND

Workman's Companion:

CONTAINING

A FULL DISPLAY AND ELUCIDATION

Of the most recent and skilful Methods, pursued by

ARCHITECTS AND ARTIFICERS,

IN THE VARIOUS DEPARTMENTS OF

CARPENTRY,	MASONRY,	PAINTING,
JOINERY,	SLATING,	GLAZING,
BRICKLAYING,	PLUMBING,	PLASTERING, &c. &c.

INCLUDING, ALSO,

NEW TREATISES

ON

GEOMETRY, THEORETICAL AND PRACTICAL, TRIGONOMETRY, CONIC SECTIONS, PERSPECTIVE, SHADOWS, AND ELEVATIONS;

A SUMMARY OF THE ART OF BUILDING;

COPIOUS ACCOUNTS OF BUILDING MATERIALS, STRENGTH OF TIMBER, CEMENTS, &c.;

A DESCRIPTION OF THE TOOLS USED BY THE DIFFERENT WORKMEN;

AN EXTENSIVE GLOSSARY OF THE TECHNICAL TERMS

PECULIAR TO EACH DEPARTMENT;

AND

THE THEORY AND PRACTICE

OF THE

FIVE ORDERS,

AS EMPLOYED IN DECORATIVE ARCHITECTURE.

By PETER NICHOLSON, Architect.

THE WHOLE ILLUSTRATED AND EMBELLISHED WITH NUMEROUS PLATES, FROM ORIGINAL DRAWINGS AND DESIGNS, MADE EXPRESSLY FOR THIS WORK, BY THE AUTHOR, AND CORRECTLY ENGRAVED, UNDER HIS IMMEDIATE INSPECTION, BY MR. W. SYMNS, AND OTHER EMINENT ARTISTS.

London:

PRINTED FOR THOMAS KELLY, 17, PATERNOSTER ROW.

1823.

PRINTED BY WEED AND RIDER, LITTLE BRITAIN, LONDON.

PREFACE.

THE WORK here respectfully submitted to the Public will be found to comprehend the PRESENT PRACTICE OF THE ART OF BUILDING, reduced to PURELY SCIENTIFIC and GEOMETRICAL PRINCIPLES, and yet explained in a manner so simple, as to be easily intelligible to any attentive Reader.

To facilitate this important object, the Author has commenced with a short Treatise on GEOMETRY, theoretical and practical, peculiarly adapted to the general object of the Work, and containing such theorems and problems only as are *absolutely necessary* to be understood by every person connected with the leading departments of the art.

And here the writer cannot refrain from adding a few words, with a view to impress upon the minds of WORKMEN engaged in the construction of buildings, whether CARPENTER, JOINER, MASON, or BRICKLAYER, the PARAMOUNT and UNSPEAKABLE IMPORTANCE of obtaining some knowledge of the PRINCIPLES of GEOMETRY; since, of all the numerous classes, concerned in mechanical arts, THEY require the most intimate acquaintance with this science.

The execution of the design of the architect is generally left to the skill of the workman; who is, of course, presumed to be fully competent to the performance of the task which he undertakes. Now, if he be not practically acquainted with the *geometrical* construction of the object to be executed, he is not only unfit for the undertaking, but, at every step

that he takes, he will manifest his ignorance and inability, and eventually overwhelm himself with confusion and disgrace. While persons of this description draw down upon themselves such merited degradation, those who, by assiduous application, have made themselves masters of the principles of geometry, and have obtained a clear and comprehensive view of the practical application of these principles, will not fail to enjoy that intellectual satisfaction which results from a successful termination of efforts, conducted with scientific skill, and crowned with general approbation; and, at the same time, open for themselves a legitimate path to that reputation which directly and naturally leads to opulence and independence.

The articles on CARPENTRY and JOINERY are treated at great length, as their superior importance demand. Indeed, it has been the chief study of the Author's life to give to these two branches the utmost degree of scientific connection and development of which they are susceptible.

In MASONRY, the artist will find an ample detail of the methods of cutting stone, illustrated by several plates, answering to most purposes which present themselves. And since the principles laid down in this Work are every where of a general tendency, the judgement of the workman will enable him to apply them wherever difficulties may occur.

The art of BRICKLAYING is but little connected with the study of geometrical lines; since the texture of bricks is such as will not admit of their being moulded to the different shapes which the ingenuity of the architect might devise. However, in order to render the present Work complete, and to obviate, as far as possible, every difficulty, several plates are introduced, illustrative of the various forms of arches, niches, &c.

Few things are more important than a clear idea of the mutual connection of the various parts of a building. The author has, therefore, introduced a section, in which he has endeavoured to show, from first principles only, the dependance which each part has upon some other.

The various trades connected with building, as PLASTERING, PAINTING, GLAZING, PLUMBING, &c. will be found to be treated of in as complete a manner as was practicable; these branches not admitting of any very scientific development.

A comprehensive Treatise on the FIVE ORDERS is subjoined to the trades accessary to building: these Orders, with their appropriate embellishments, form the basis and superstructure of architectural decoration. The parts of the Orders are drawn on a scale which speaks to the eye, and renders all farther detail unnecessary. The parts are given in modules and minutes; this being the best mode of exhibiting their proportions, so as to be most readily and clearly comprehended by the workman and student.

In order to increase the utility of the Work, to the BUILDER and CONTRACTOR, a select series of designs, in the modern style, accommodated to the various ranks of society, have been introduced; and, for the use of the ARCHITECTURAL STUDENT, that no accomplishment which might facilitate the operations of the draughtsman, or furnish the designer with more correct ideas, or more extensive views, may be wanting, the RULES of PROJECTION, and the PRINCIPLES of PERSPECTIVE, are presented; and in the most familiar and simple manner in which the subject could be conceived.

The Work concludes with a copious GLOSSARY of the most useful terms employed by architects and builders.

On the whole, the following Treatise will be found to contain a much greater variety of subjects than any similar work; and, in the method of treating the various articles, the studious reader will discover many things entirely new. Thus, for example, in the designs for roofs, several modes are brought forward, for the first time, interesting, both with respect to the disposition and joining of the timbers; and the examples which are given

will be found of the greatest utility to the practical builder, in regulating his ideas with respect to any design under consideration, however much it may differ from any of the forms exhibited here.

The schemes or diagrams are proportioned in their size to their probable utility; and the strictest regard has been paid to giving to all the parts of each figure their respective and just proportions.

Finally, from the important information collected, the natural arrangement adopted, and the numerous and valuable illustrations exhibited in the course of this Work, the Author flatters himself that he will be found to have rendered an important service to a numerous and highly meritorious class of his fellow-subjects; whilst even the most inattentive observer cannot but acknowledge that the Publisher has spared no expense to render the Work deserving of extensive patronage and general approbation. The grand principle of the undertaking is obvious: it is equally calculated to instruct the untaught, and to assist the intelligent; to promote a generous emulation, and at once to incite and satisfy enquiry into the elements and practice of those branches of science, than which no others are more conducive to the comfort and happiness of mankind.

P. NICHOLSON.

London, 1822.

THE NEW

PRACTICAL BUILDER, &c.

CHAPTER I.

THE ELEMENTS OF GEOMETRY.

GEOMETRY is a science which considers the properties of lines and angles, as formed according to some certain law; as, also, the construction of all manner of figures, according to given *data.*

It is divided into two branches; one of which considers the relations, positions, and properties, of lines, so as to render a proposition clear to the understanding without the aid of compasses or other instruments; being demonstrated, by a continued chain of reasoning, from certain principles previously established and laid down as axioms; so that the conclusions from one truth become part of the *data* for the proof of a succeeding proposition. This, which is called the Theory of Geometry, is fully explained by EUCLID, in his celebrated "ELEMENTS," which have served as the basis of all succeeding treatises on the subject: and so much of those Elements as may be required in the practice of Architecture will be found included in the present work.

The other branch of geometry is entirely *practical,* and may be acquired without the theory, according to the directions hereafter given; although with a knowledge of the reasons of the rules it will be more satisfactory. It is this practical branch that enables the architect to regulate his designs, and the artizan to construct his lines, so as to enable him to execute the work. Without the aid of this branch of knowledge, the workman will be unfit for

any undertaking whatever; and, so long as he is ignorant of the methods of geometrical construction, he must remain under the control and direction of a superior in his own class.

The definitions and problems, which follow, are calculated to instruct the uninformed mechanic, and will qualify him for proceeding to the remaining parts of this treatise, wherein it will be found that the application of this branch of science is absolutely necessary.

The uses of Geometry are not confined to Carpentry and Architecture: Astronomy, Navigation, Perspective, and numerous other branches, are entirely dependent upon it. "It conducts the soldier in the field, and the seaman on the ocean; it gives strength to the fortress, and elegance to the palace." In short, there is no mechanical profession that does not derive considerable advantage from it. One workman is superior to another, in proportion to his knowledge of the subject we are now commenting upon, and which we are about to explain.

The Terms are here as clearly defined as the nature of the subject will admit, and the Problems are put in a regular succession; so that nothing is introduced, in any problem, as taken for granted, but what has been explained in some problem previously given. This selection, though not very numerous, is sufficient to enable the student to proceed with the remaining parts of the work, to which it is specially adapted: and every attention has been paid to divest the diagrams of superfluous lines, without rendering them less intelligible.

GEOMETRIC DEFINITIONS.

1. A Point is considered as that which has position without magnitude. *Practically,* a Point is the smallest visible mark upon a surface, as at *figure* 1, *plate* I.

2. A Line is considered as length, without breadth or thickness; having extension only in one direction, as *figures* 2 and 3, (*plate* I,) which may be conceived to be made by the trace of a point, pen, or pencil.

3. A RIGHT or STRAIGHT LINE is that which lies evenly between its extremes or ends. If two straight lines coincide in two points, all the intermediate points will coincide also.

Thus, *fig*. 3, (*pl.* I,) represents a straight line, and *fig*. 2, a curve, or crooked line; the latter may be formed either by regular inflexions, or portions of straight lines, or both.

4. A SUPERFICIES, or SURFACE, is that which is considered as having length and breadth without depth.

Thus the outward parts of any body, which are exhibited to the eye, are termed the *superficies* of that body.

5. A PLANE SUPERFICIES or PLANE SURFACE, is that on which a straight line, drawn through any given point, in any position, will coincide.

6. A PLANE FIGURE, or DIAGRAM, or SCHEME, is the representation of any thing on a plane surface, by means of lines. When the lines are straight, the figure is said to be *rectilineal.*

7. An ANGLE is a space between two lines meeting in a point. A PLANE RECTILINEAL ANGLE is the space between two straight lines so meeting.

Thus, *fig*. 4, (*pl.* I,) is a *plane rectilineal angle.*

8. Two straight lines are said to *converge,* when they meet each other, if *produced* or continued; as in *fig*. 5.

9. When one straight line stands upon another, and makes the angles on each side equal to each other, each of the equal angles is called a RIGHT ANGLE, and the line which stands upon the other is called a *perpendicular* to that other line.

Thus, in *fig*. 6, (*pl.* I,) if the line CD stand upon AB, and make the angles on both sides of CD equal; each of these angles is a *right angle.* In *fig*. 7, the line CD does not make the angles on each side of it equal to each other: in this case, CD is said to stand at *oblique angles* to AB; and in the former case, *fig*. 6, CD is said to stand at right angles to AB.

10. An ACUTE ANGLE is that which is *less* than a right angle.

11. An OBTUSE ANGLE is that which is *greater* than a right angle.

In *fig.* 7, as CD makes the angles on each side of it unequal, one of them must be greater than the other: the greater must, therefore, be an *obtuse angle,* and the less an *acute angle.* And, as the space around the point C is the same, whatever be the position of the line CD, with respect to AB, what the one angle has in excess above the right angle, the other will have as much in defect.

Figure 8, (*pl.* I,) is an *acute angle; fig.* 9, a *right angle;* and *fig.* 10, an *obtuse angle.*

12. A Plane Triangle is a space inclosed by three straight lines.

Thus, *figures* 11, 12, 13, and 14, are *triangles.*

13. A Right-angled Triangle is that which has one right angle.

Thus, *fig.* 11 is a *right-angled triangle.*

14. An Acute-angled Triangle is that which has all its angles *acute;* as *figures* 12 and 13.

15. An Obtuse-angled Triangle is that which has one *obtuse* angle; as *fig.* 14.

16. An Equilateral Triangle is that which has all its sides *equal;* as *fig.* 12.

17. An Isosceles Triangle is that which has two equal sides; as *fig.* 13.

18. A Scalene Triangle is that which has no two of its sides equal; as *fig.* 14.

19. Parallel Lines are lines on the same plane, which cannot meet, though produced or continued ever so far from each extremity (*fig.* 15.)

20. A Parallelogram is a figure whose opposite sides are *parallel.*

Thus, *figures* 16, 17, 18, and 19, are *parallelograms.*

21. When the parallelogram has one of its angles a right-angle, it is called a rectangle. Thus, *figures* 16 and 17 are *rectangles.*

22. When the sides of the rectangle are equal, it is called a square.

Thus, *fig.* 16 is a *square.*

23. When the two adjacent sides are unequal, the rectangle is called an oblong; as *fig.* 17.

24. When only two opposite angles of a parallelogram are equal, it is called a RHOMBUS; as *figures* 18 and 19.

25. When two adjacent sides of a rhombus are equal, it is called a RHOMBOID (pron. *rhom-bo-id)*; as *fig.* 19.

26. Every figure, inclosed by four straight lines, is called a QUADRANGLE or QUADRILATERAL. Thus, *figures* 16, 17, 18, 19, 20, and 21, are *quadrilaterals.*

27. When all the sides of a quadrilateral are unequal, it is called a TRAPEZIUM.

28. When two sides of the trapezium are parallel, it is called a TRAPEZOID; as *fig.* 21.

29. Equilateral and equiangular figures, contained by more than four straight lines, are called REGULAR POLYGONS.

30. A regular polygon of *five* sides, is called a PENTAGON; as *fig.* 22.

31. A regular polygon of *six* sides, is called a HEXAGON; as *fig.* 23.

32. A regular polygon of *seven* sides, is called a HEPTAGON; as *fig.* 24.

33. A regular polygon of *eight* sides, is called an OCTAGON; as *fig.* 25, and so on.

The words *enea, deca, undeca, dodeca,* having the termination *gon* subjoined, signify regular polygons of nine, ten, eleven, and twelve, sides. Other polygons are commonly expressed as such, with the number of sides.

34. A CIRCLE is a plain figure, contained under one line only, which is called its *circumference.* From the circumference, straight lines, called *radii,* being drawn to a certain point within the figure, are equal.

35. The point to which the equal lines from the circumference are drawn, is called the CENTRE of the circle. Thus, in *fig.* 26, *c* is the *centre,* and *c d* the *radius* of the circle *a b d.*

36. The DIAMETER of a circle is a straight line, drawn through the centre, and terminated by the circumference; as the line *a b, fig.* 27.

37. A CHORD of a circle is a straight line, drawn through the circle, and terminated by the circumference. Thus the line *a b, fig.* 28, is a *chord;* and *a b, fig.* 27, is a *chord passing through the centre.*

38. A SEMI-CIRCLE is the half of a circle, terminated by a diameter and the semi-circumference. Thus, in *fig.* 27, the diameter *a b* divides the circle into two semi-circles.

39. A SEGMENT of a circle is a portion cut off by a chord, and the part of the circumference intercepted by the chord. Thus, *a b c*, *figures* 28 and 29, are *segments*; and *fig.* 30, though a semi-circle, is still a segment, terminated by the diameter, instead of a lesser chord.

40. A SECTOR of a circle is the portion contained by two radii and the intercepted part of the circumference. Thus, *a b c*, *fig.* 31, is the *sector* of a circle.

41. The QUADRANT of a circle is a sector contained by two radii, at a right-angle with each other, and the intercepted part of the circumference; as, *a b c*, in *fig.* 32.

42. An ARC of a circle is any portion of its circumference.

43. The ALTITUDE of a figure is a straight line drawn from the vertical angle, perpendicular to the opposite side, or to the opposite side produced or continued. Thus, CD, *fig.* 33, is the altitude of the triangle ABC, drawn from the vertical angle C to the opposite side AB produced to D.

44. NOTATION.

When several angles unite at a point, each angle is indicated by *three letters*, the middle letter denoting the *angular point*, and the others the sides containing that angle. Thus, in *fig.* 34, ABC, ABD, ABE, the middle letter B indicates the angular point: in the first, AB, BC, the two sides; in the second, AB, BD, the sides; and, in the third, AB, BE, the sides.

45. EXPLANATION OF TERMS.

An AXIOM is a self-evident truth.

A THEOREM is a truth which becomes evident by a process of reasoning called a *demonstration*.

A Problem is a thing required to be done, or a question proposed for solution.

A Lemma is a truth premised to facilitate either the demonstration of a theorem, or the solution of a Problem.

A Proposition is the common name of a Theorem or Problem.

A Corollary is a consequence or deduction which follows from a Proposition.

A Scholium is an explanatory remark upon one or more preceding Proposition or Propositions.

An Hypothesis is a *supposition* made either in the enunciation of a Proposition, or in the course of a demonstration.

46. AXIOMS.

1. Things which are equal to the same thing, or things, are equal to one another.
2. If equals be added to equals, the wholes will be equal.
3. If equals be taken from equals, the remainders will be equal.
4. If equals be added to unequals, the wholes will be unequal.
5. If equals be taken from unequals, the remainders will be unequal.
6. Things which are double of the same thing, are equal to one another.
7. Things which are halves of the same thing, are equal to one another.
8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.
9. The whole is greater than its part.
10. Only one straight line can be drawn from one point to another.
11. Two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another.

47. POSTULATES, or DEMANDS.

1. Let it be granted that a straight line may be drawn from any one point to any other.

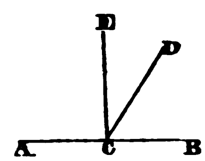
2. That a terminated straight line may be *produced*, or continued, to any length.

3. That a circle may be described from any centre, and at any distance from that centre, or with any radius.

THEOREMS.

THEOREM 1.

48. Any straight line, CD, which meets another straight line, AB, makes with it two adjacent angles, ACD, BCD; which, taken together, are equal to two right angles.*

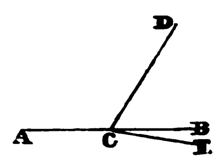


At the point C, let the straight line CE be drawn, perpendicular to AB. The angle ACD is the sum of the angles ACE and ECD; therefore ACD+DCB shall be the sum of the three angles ACE, ECD, DCB, (*Axiom* 2, page 15). Now the angle ACE is a *right angle*, and the sum of the angles ECD, DCB, make a right angle; therefore the sum of the two angles ACD, BCD, is equal to two right angles.

49. COROLLARY.—If one of the angles ACD, BCD, is a right angle, the other is, also, a right angle.

THEOREM 2.

50. If the sum of two adjacent angles, ACD, DCB, be equal to two right angles, the exterior sides form one continued straight line.



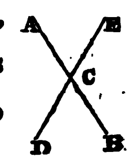
For, if CB is not the continuation of AC, let CE be its continuation; then the sum of the angles ACD, DCE, is equal to two right angles, (*theorem* 1,) but, by hypothesis, the sum of the angles ACD, DCB, is equal to two right angles; therefore the two angles ACD, DCE, is equal to the two angles

* The signs used in Algebraic Notation are explained hereafter; but, as several may previously occur, it may here be noticed that + (*plus*) signifies *more*, or one quantity or thing added to another: The sign − (*minus*) signifies *less*, or one quantity subtracted from another: = means is, or are, equal to: × into, or multiplied by: ÷ divided by, as 30÷3 or $\frac{30}{3}$=10.

ACD, DCB (*Ax.* 1, page 15); and, taking from each of these equals the angle ACD, there will remain the angle DCE, equal to DCB, a part equal to the whole, which is impossible (*Ax.* 9, page 15).

THEOREM 3.

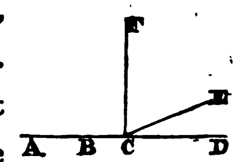
51. If two straight lines, AB, DE, cut one another, the opposite angles shall be equal to one another.



For, since DE is a straight line, the sum of the two angles ACD, ACE, is equal to two right angles (*theorem* 1); and, because AB is a straight line, the sum of the angles ACE, ECB, is equal to two right angles (*theorem* 1); therefore the sum of the angles ACD, ACE, is equal to the sum of the angles ACE, ECB; and, taking away from each the common angle ACE, there will remain the angle ACD, equal to the vertical opposite angle ECB.

THEOREM 4.

52. Two straight lines, which have two common points, coincide entirely throughout their whole extent.

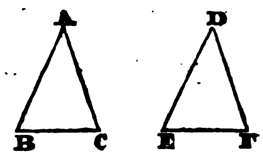


Let A and B be the two common points; in the first place, the two lines can make but one from A to B, (*Ax.* 10, *p.* 15). If it were possible that they could separate, let C be the point of separation, and let us suppose that one of them takes the direction CD, and the other CE.

At the point C suppose CF to be drawn, perpendicular to AC; then, because ACD is, by hypothesis, a straight line, the angle FCD is a right angle; (*Def. art.* 9;) in like manner, because ACE is supposed to be a straight line, the angle FCE is a right angle; therefore the angles FCD, FCE, are equal; but this is impossible (*Ax.* 9, *page* 15); therefore the two straight lines, which have two common points, A and B, cannot separate, but must form one continued line.

THEOREM 5.

53. Two triangles are equal when, in the one, an angle and the two sides which contain it are equal, in the other, to an angle and the two sides which contain it.

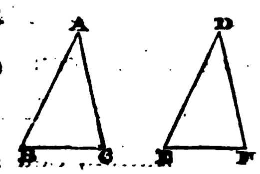


Let the angle A be equal to the angle D, the side AB equal to DE, and the side AC equal to DF; then the triangles ABC, DEF, shall be equal.

Suppose the triangle ABC to be placed upon the triangle DEF, so that AB may be upon DE; then, because the angles A and D are equal, AC will fall upon DF; and, because AB is equal to DE, and AC equal to DF, the point B will coincide with E, and C with F: therefore, the base BC will coincide with the base EF (*theorem* 4); and, since the sides of the triangles coincide, the other two angles must also coincide; that is, they must be equal to each other.

THEOREM 6.

54. Two triangles are equal when a side and two adjacent angles of the one are respectively equal to a side and two adjacent angles of the other.



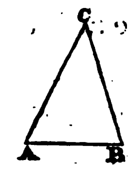
Let the side BC be equal to the side EF, the angle B equal to the angle E, and the angle C equal to the angle F, the triangles shall be equal.

For, suppose the triangle ABC to be placed upon the triangle DEF, so that their bases, BC and EF, may coincide; then, because the angles B and E are equal, the straight line BA will fall upon ED; and, because the angles C and F are equal, the straight line CA will fall upon FD: therefore, the three sides of one triangle will coincide with the three sides of the other; and, consequently, the triangles themselves will be equal: and, since therefore the sides of the triangles coincide, the corresponding angles will be equal.

THEOREM 7.

55. Any two sides of a triangle are together equal to more than a third side.

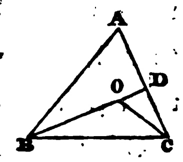
For, in the triangle, ABC, the straight line BC is the shortest line that can be drawn from B to C; therefore the sum of the two sides, BA, AC, is greater than BC.



THEOREM 8.

56. If from any point, as O, within a triangle, ABC, there be drawn two straight lines, OB, OC, one to each extremity of any side, as BC, their sum is less than the sum of the other two sides of the triangle.

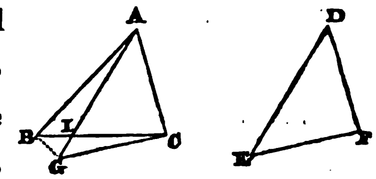
Produce BO till it meet AC in D; the line OC is less than the sum of the two lines OD, DC (*theorem* 7); and, adding to these unequals the line BO, the sum of the two lines, BO, OC, is less than the sum of the three lines BO, OD, DC (*ax.* 4, *p.* 15); that is, the sum of the two lines BO, OC, is less than the sum of the two lines BD, DC.



In like manner, BD is less than the sum of the two lines BA, AD; and, adding DC to these unequals, the sum of the two straight lines, BD, DC, is less than the sum of the three straight lines BA, AD, DC; that is, the two straight lines BD, DC, are less than the two straight lines BA, AC; but the two straight lines BO, OC, have been shown to be less than the two straight lines BD, DC; and, therefore, much less is the sum of the two straight lines BO, OC, than that of the two sides BA, AC, of the triangle ABC.

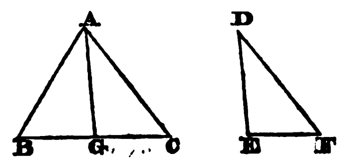
THEOREM 9.

57. If any two sides AB, AC, of a triangle, ABC, are equal to two sides DE, DF, of another triangle DEF, each to each, and if the angle BAC, contained by the sides, AB, AC, be greater than the angle EDF, contained by the sides ED, DF, the base BC of the triangle which has the greater angle shall be greater than the base EF of the other triangle.

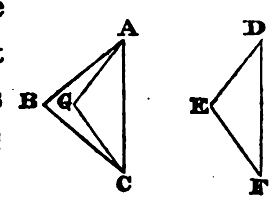


Make the angle CAG equal to D, take AG equal to DE or AB, and join CG; and because the two triangles CAG, DEF, have an angle of the one equal to an angle of the other, and the sides which contain these angles are equal, CG shall be equal to EF (*theorem* 5). Now there may be three cases, according as the point G falls without the triangle ABC, or on the side BC, or within the triangle.

Case 1.—Because GC is less than the sum of the two straight lines GI, IC; and AB less than the sum of the two straight lines AI, IB: therefore, the sum of the two straight lines GC, AB, is less than the sum of the four straight lines GI, IC, AI, IB; that is, the sum of the two straight lines GC, AB, is less than the sum of the two straight lines AG, BC; but AG is equal to AB, therefore GC is less than BC; but GC = EF, therefore EF is less than BC.



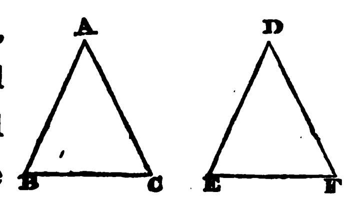
Case 2.—If the point G fall on BC, it is evident that GC, or its equal EF, is less than BC.



Case 3.—Lastly, if the point G fall within the triangle ABC, by *theorem* 8, we have the sum of the two straight lines AG, GC, less than the sum of the two straight lines AB, BC; but since AB is equal to AG, we shall have GC less than BC; and, consequently, EF less than BC.

THEOREM 10.

58. One triangle is equal to another, when the three sides of the first are respectively equal to the three sides of the second.



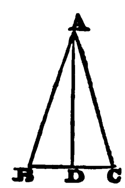
Let the side AB be equal to DE, AC equal to DF, and BC equal to EF; then shall the angle A be equal to the angle D, the angle B equal to the angle E, and the angle C equal to the angle F. For, if the angle A were greater than D, then, as the two sides AB, AC, are equal to the two sides DE, DF, each to each, it would follow (*theorem* 9) that the side BC

would be greater than EF; and, if the angle A were *less* than the angle D, BC would be less than EF; therefore, the angle A can neither be greater nor less than the angle D; the angle A must therefore be equal to the angle D. In like manner, it may be proved that the angle B is equal to E, and C equal to F.

59. Corollary.—Whence it appears that, in two equal triangles the equal angles are opposite to the equal sides; for the equal angles A and D are opposite to the equal sides BC and EF.

THEOREM 11.

60. The angles opposite to the equal sides of an isosceles triangle are equal.



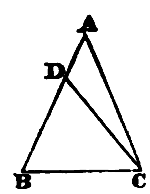
Let the side AB be equal to AC; then shall the angle C equal the angle B. For, suppose AD to be drawn from the vertex A to the middle point D, of the base BC; then the two triangles ADB, ADC, will have the two sides AB, BD, of the one equal to the two sides AC, CD, of the other, each to each; and AD is common to both: therefore the angle B shall be equal to the angle C.

61. Corollary 1.—Hence every equilateral triangle is also equiangular.

62. Corollary 2.—A straight line drawn from the vertex of an isosceles triangle to the middle of the base will bisect the vertical angle, and be perpendicular to the base.

THEOREM 12.

63. If two angles of a triangle be equal, the opposite sides shall be equal, and the triangle shall be isosceles.



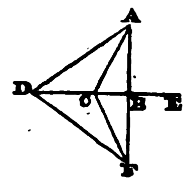
Let the angle ABC be equal to ACB, the side AC shall be equal to the side AB.

For, if the two sides AB, AC, are not equal, let AB be greater than AC, and from BA cut off BD, equal to CA, and join CD; the angle DBC is, by hypothesis, equal to the angle ACB, and the two sides DB, BC, are equal to the two sides AC, CB; therefore the triangle DBC is equal

to the triangle ACB, the less to the greater, which is impossible (*ax.* 9, *p.* 15); therefore AB cannot be unequal to AC, but must be equal to it.

THEOREM 13.

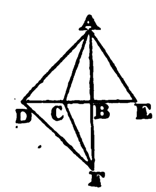
64. From a point A, without a straight line DE, only one perpendicular can be drawn to that line.



For, suppose it were possible to draw AB, AC, perpendicular from the same point A, upon the straight line DE; produce one of them, AB, to F, so that BF may be equal to AB, and join FC; and, because AB is equal to BF, and BC is common to the two triangles ABC, FBC, and the angles ABC and FBC are equal; the angle ACB is equal to FCB (*theorem* 5); therefore AC and CF must be a continued line (*theorem* 2); and so, through the two points A, F, two straight lines, AF and ACF, may be drawn, that do not coincide; which is impossible: and, therefore, it is equally impossible that two perpendiculars can be drawn from the same point to the same straight line.

THEOREM 14.

65. Of all the lines that can be drawn from a given point A, to a given straight line DE, the perpendicular is the shortest; and of the other lines, that which is nearer the perpendicular is less than that which is more remote; and those two lines, on opposite sides, and at equal distances, from the perpendicular, are equal.



Produce the perpendicular, so that BF may be equal to AB, and draw the straight lines AC, AD, and AE, to meet DE in C, D, and E, and join FC, FD, &c.

The triangles BCF and BCA are equal (*theorem* 5); for BF is equal to BA, and BC common; therefore CF is equal to CA. Now AF is less than AC+CF (*theorem* 7); therefore, taking the halves, AB is less than AC; that is, the perpendicular is the shortest line that can be drawn from A to DE.

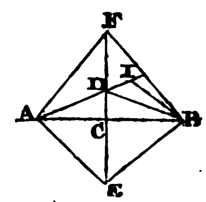
Next, suppose BE equal to BC; then the triangles ABE and ABC will be equal (*theorem* 5), for they have BA common, and the angles ABE and ABC

equal; therefore AE is equal to AC; that is, two oblique lines equally distant from the perpendicular, on opposite sides, are equal.

In the triangle ADF, the sum of AC and CF is less than the sum of AD and DF (*theorem* 8); therefore AC, the half of AC+CF, is less than AD, the half of AD+DF; that is, the oblique line, which is farther from the perpendicular, is greater than that which is nearer to it.

THEOREM 15.

66. If through the point C, the middle of the straight line AB, a perpendicular be drawn to that line, every point in the perpendicular is equally distant from the extremities of the line AB, and every point out of the perpendicular is unequally distant from these extremities.

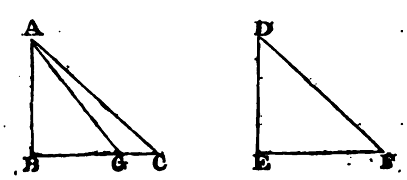


Because AC is equal to BC, the two oblique lines AD, BD, which are equally distant from the perpendicular, are equal (*theorem* 14). The same is also true of the two oblique lines AE, EB, and of the two oblique lines AF, FB, &c. Therefore, every point in the perpendicular is equally distant from the ends of the line.

Let I be a point out of the perpendicular. If IA, IB, be joined, one of them will cut the perpendicular in D; therefore, drawing DB, we have DB equal to DA: but IB is less than ID+DB, and ID+DB is equal to ID+DA equal to IA; therefore IB is less than IA: that is, any point out of the perpendicular is unequally distant from the extremities A and B.

THEOREM 16.

67. Two right-angled triangles are equal if the hypothenuse and a side of the one be equal to the hypothenuse and a side of the other.

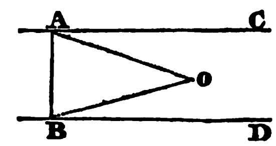


Let the hypothenuse (or longest side) AC be equal to the hypothenuse DF, and the side AB equal to the side DE, and the right-angled triangle ABC shall be equal to the right-angled triangle DEF.

The proposition will be evidently true if it can be proved that BC is equal to EF (*theorem* 10). Let us suppose, if it be possible, that these sides are unequal, and that BC is the greater. Take BG equal to EF, and join AG. The triangles ABG and DEF, having AB equal to DE, and BG equal to EF, by hypothesis, and also having the angle ABG equal to DEF, they will be equal (*theorem* 5): therefore AG is equal to DF; but DF is equal to AC; therefore AG is equal to AC: that is, two oblique lines, one more remote from the perpendicular than the other, are equal; which is impossible (*theorem* 15): therefore, BC is not unequal to EF, and hence the triangle ABC is equal to the triangle DEF.

THEOREM 17.

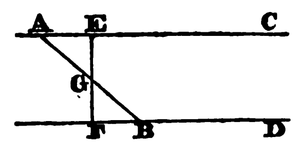
68. Two straight lines perpendicular to a third are parallel.



For, if the straight lines AC, BD, be not parallel, they will meet on one side or the other of the line AB; let them meet in O; then AC and OB are both perpendicular to AB, from the same point O; which is impossible (*theorem* 13).

THEOREM 18.

69. If two straight lines, AC and BD, make, with a third, AB, the sum of the two interior angles CAB, ABD, equal to two right angles, these two straight lines are parallel.

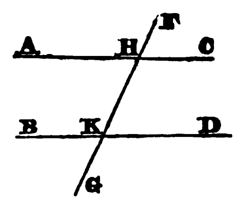


From G, the middle of AB, draw EGF, perpendicular to AC: then, since the sum of the angles ABD, ABF, is equal to two right angles (*theorem* 1), and, by hypothesis, the sum of the two angles ABD, BAC, is also equal to two right angles; therefore the two angles ABD, ABF, are together equal to the sum of the two angles ABD, BAC; and, taking away the common angle ABD, there remains the angle ABF = BAC; that is, GBF equal to GAE. But the angles BGF and AGE are also equal (*theorem* 3); and, since BG is equal to GA, therefore the triangles BGF and AGE, having a side and two adjacent angles of the one equal to a side and two adjacent angles of the other, are equal (*theorem* 6),

and the angle BFG is equal to AEG; but AEG is, by construction, a right angle; therefore, BFG is also a right angle; and, since GEC is a right angle, the straight lines EC and FD are perpendicular to EF, and are, therefore, parallel to each other (*theorem* 17).

THEOREM 19.

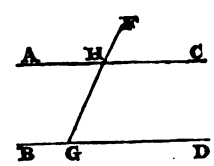
70. If two straight lines, AC, BD, make with a third, HK, the alternate angles, AHK and HKD, equal, the two lines are parallel.



For, adding KHC to each of the angles AHK, HKD, the sum of the angles AHK, KHC, is equal to the sum of the angles HKD, KHC; but the angles AHK, KHC, are together equal to two right angles; therefore, also, the angles HKD, KHC, are also equal to two right angles; and, consequently, AC is parallel to BD (*theorem* 18).

THEOREM 20.

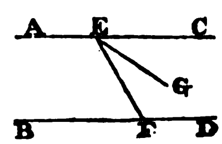
71. If two straight lines, AC, BD, are cut by a third, FG, so as to make the exterior angle, FHC, equal to the interior and opposite angle, HGD, on the same side, the two lines are parallel.



For, since the angle FHC is equal to the angle AHG, and since, when AC is parallel to BD, the angle AHG is equal to HGD (*theorem* 19), therefore the angle FHC is equal to HGD.

THEOREM 21.

72. If a straight line, EF, meet two parallel straight lines, AC, BD, the sum of the inward angles CEF, EFD, on the same side, will be equal to two right angles.



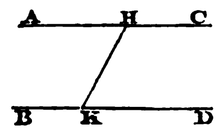
For, if not, suppose EG to be drawn through E, so that the sum of the angles GEF and EFD may be two right angles; then EG will be parallel to BD (*theorem* 18); and thus, through the same point E, two straight lines, EG, EC, are drawn, each parallel to BD; which is impossible (*ax.* 11, *p.* 15); therefore no straight

line that does not coincide with AC, is parallel to BD; wherefore the straight line AC is parallel to BD.

73. COROLLARY.—If a straight line is perpendicular to one of two parallel straight lines, it is also perpendicular to the other.

THEOREM 22.

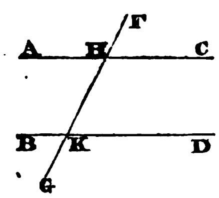
74. If a straight line, HK, meet two parallel straight lines, AC, BD, the alternate angles, AHK, HKD, shall be equal.



For, the sum of the angles CHK, HKD, is equal to two right angles; and the sum of the angles BKH, AHK, is also equal to two right angles; therefore the angle HKD must be equal to AHK.

THEOREM 23.

75. If a straight line FG, cut two parallel straight lines, AC, BD, the exterior angle, FHC, is equal to the interior and opposite angle HKD.

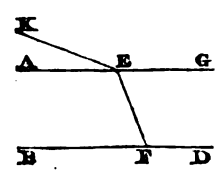


For, since the angle FHC is equal to the angle AHK (*theorem* 3), and the angle AHK equal to the angle HKD; therefore the angle FHC is equal to the angle HKD.

THEOREM 24.

76. If a straight line, EF, meet two other straight lines, EG, FD, and make the two interior angles, EFD, FEG, on the same side, less than two right angles, the lines EG, FD, meet, if produced, on the side of EF, on which the angles are less than two right angles.

For, if they do not meet on that side, they are either parallel, or else they meet on the other side. Now they cannot be parallel, for then the two interior angles would be equal to two right angles, instead of being less. Again, to show that they cannot meet on the other side, suppose EA to be parallel to DFB; then, because the sum of the angles EFD, FEG, is, by hypothesis less than two right angles, that is, less than the sum of the two angles,

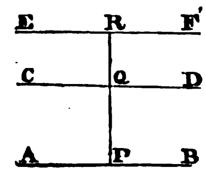


FEK, FEG (*theorem* 1), and EFD is equal to FEA (*theorem* 20); therefore the sum of the two angles FEA, FEG, is less than the sum of the two angles FEK, FEG; and, taking FEG from both, FEA is less than FEK: hence, FB and EK must be on opposite sides of EA; and, therefore, can never meet.

The truth of this proposition is assumed as an axiom in the Elements of Euclid, and made the foundation of parallel lines.

THEOREM 25.

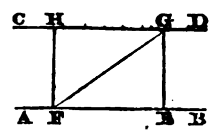
77. Two straight lines, AB, CD, parallel to a third, EF, are parallel to one another.



Draw the straight line PQR, perpendicular to EF. Because AB is parallel to EF, the line PR shall be perpendicular to AB; and, because CD is parallel to EF, the line PR is also perpendicular to CD: therefore AB and CD are perpendicular to the same straight line PQ; hence they are parallel (*theorem* 17).

THEOREM 26.

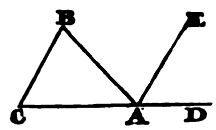
78. Two parallel straight lines are every where equally distant.



Let AB, CD, be two parallel straight lines. From any points, E and F, in one of them, suppose perpendiculars EG, FH, to be drawn; these, when produced, will meet the others at right angles, in H and G. Join FG; then, because FH and EG are both perpendicular to AB, they are parallel (*theorem* 17); therefore, the alternate angles, HFG, FGE, which they make with FG are equal (*theorem* 22): and, because AB is parallel to CD, the alternate angles, GFE, FGH, are also equal; therefore the two triangles GEF, FHG, have two angles of the one equal to two angles of the other, each to each; and the side FG, adjacent to the equal angles, common; the triangles are therefore equal (*theorem* 6); and FH is equal to EG; that is, any two points, F, E, on the one of the lines, are equidistant from the other line.

THEOREM 27.

79. In any triangle, if one of the sides be *produced*, the exterior angle is equal to both the interior and opposite angles; and the three interior angles are equal to two right angles.



Let ABC be a triangle; produce any one of its sides, AC towards D; and, from the point A, let AE be drawn, parallel to BC; and, because of the parallels CB and AE, and the angle EAD = C, and the angle EAB = B (*theorems* 22, 23); therefore the sum of the two angles, EAD, EAB, is equal to the sum of the two angles C and B; that is, since the angle BAD is equal to the sum of the two angles BAE, EAD, the angle BAD is equal to the sum of the angles B, C. Hence the outward angle is equal to the sum of the inward opposite angles.

Again, because the angle BAD is equal to the sum of the angles B and C, add to each the angle BAC, and the sum of the two angles BAC, BAD, will be equal to the sum of the three angles BAC, B, C, or the three angles of the triangle; but the sum of the two angles BAC, BAD, is equal to two right angles (*theorem* 1); therefore the sum of the three angles of a triangle is equal to two right angles.

80. Corollary 1.—If two angles of one triangle be equal to two angles of another triangle, each to each, the third angle of the one shall be equal to the third angle of the other, and the triangles shall be equi-angular.

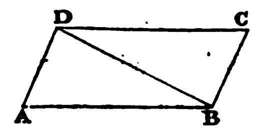
81. Corollary 2.—A triangle can have only one right angle.

82. Corollary 3.—In any right-angled triangle the sum of the two acute angles is equal to a right angle.

83. Corollary 4.—In an equilateral triangle, each of the angles is one-third of two right angles.

THEOREM 28.

84. The opposite sides of a parallelogram are equal, as well as the opposite angles.



Draw the diagonal BD. The triangles ADB, DBC, have the common side DB; also, because of the parallels, AB, CD, the angle ABD is equal to CDB (*theorem* 22); and, because of the parallels AD, BC, the angle ADB is equal to DBC; therefore the triangles (*theorem* 6) and the sides AB, DC, which are opposite the equal angles, are equal. In like manner AD and BC are equal; therefore the opposite sides of the parallelogram are equal.

Again, from the equality of the triangles, it follows, that the angle A is equal to the angle C; and it has been shown that the angles ADB, BDC, are respectively equal to the angles CBD, DBA; therefore the whole angle ADC is equal to the whole angle ABC, and thus the opposite angles are equal.

85. Corollary.—Two parallels, AB, CD, comprehended between two other parallels, AD, BC, are equal.

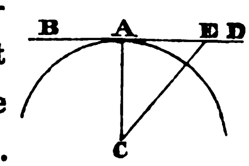
THEOREM 29.

86. If the opposite sides of a quadrilateral be equal, the figure is a parallelogram.

For, drawing the diagonal BD, (as above,) the triangles ABD, BDC, have the three sides equal, each to each; therefore the angle ADB, opposite to the side AB, is equal to the angle CBD, opposite to the side CD (*theorem* 10); hence the side AD is parallel to BC (*theorem* 19). For the like reason AB is parallel to CD: therefore the quadrilateral, ABCD, is a parallelogram.

THEOREM 30.

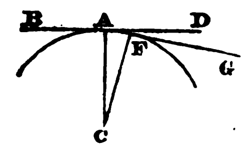
87. A straight line, BD, drawn perpendicular to the extremity of a radius, CA, is a *tangent* to the circumference.



For every oblique line, CE, is longer than the perpendicular CA (*theorem* 15); therefore the point E must be without the circle; and since this is true of every point in the line BD, except the point A, the line BD is a *tangent* (*def.* 9, *p.* 11).

THEOREM 31.

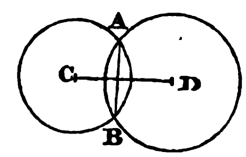
88. Only one tangent can be drawn from a point, A, in the circumference of a circle.



Let BD be a tangent at A, in the circumference, described with the radius CA; and let AG be another tangent, if possible; then, as CA would not be perpendicular to AG, another line, CF, would be perpendicular to AG, and so CF would be less than CA (*theorem* 14); therefore F would fall within the circle, and AF, if produced, would cut the circumference.

THEOREM 32.

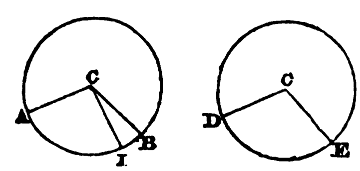
89. If two circumferences cut each other, the straight line which passes through their centres shall be perpendicular to the chord which joins the points of intersection, and shall divide it into two equal parts.



For the line AB, which joins the points of intersection, being a common chord to the two circles; if, through the middle of this chord, a perpendicular be drawn, it will pass through the points C, D, the centres of the two circles. But only one line can be drawn through two given points; therefore the straight line which passes through the centres is a perpendicular to the middle of the common chord.

THEOREM 33.

90. In the same circle, or in equal circles, equal angles ACB, DCE, at the centre, intercept equal arcs, AB, DE, on the circumference; and conversely, if the arcs AB, DE, be equal, the angles ACB and DCE are also equal.



If the angle ACB be equal to DCE, these two angles may be placed on each other; and, as their sides are equal, the point A will fall on D, and the point B on E; but then the arc AB must also fall on DE; for, if the two arcs did not coincide, there would be, in one or the other, points unequally distant from the centre; therefore the arc AB is equal to DE.

Next, if the arc AB be equal to DE, the angle ACB shall be equal to DCE; for, if they are not equal, let ACB be the greater, and take ACI equal to DCE; then, by what has been demonstrated, AI is equal to DE; but, by hypothesis, the arc AB is equal to DE; therefore the arc AI is equal to AB, which is impossible: therefore the angle ACB is equal to DCE.

THEOREM 34.

91. An angle, ACB, at the centre of a circle, is double of the angle at the cirumference, upon the same arc, AB.

Fig. 1.

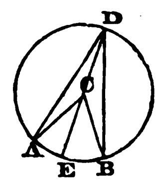
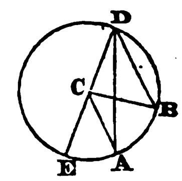


Fig. 2.



Draw DC, (*fig.* 1,) and produce it to E. First, let the angle at the centre be within the angle at the circumference, then the angle ACE is equal to the sum of the angles CAD, CDA (*theorem* 27); but, because CA is equal to CD, the angle CAD is equal to CDA (*theorem* 11); therefore the angle ACE is equal to twice the angle CDA. By the same reason the angle BCE is equal to twice the angle CDB; therefore the whole angle ACB is double the whole angle ADB.

Next, let the angle at the centre (*fig.* 2,) be without the angle at the circumference. It may be demonstrated, as in the first case, that the angle ECB is equal to twice the angle EDB, and that the angle ECA, a part of the first, is equal to twice EDA, a part of the second; therefore, the remainder, ACB, is double the remainder ADB.

THEOREM 35.

92. The angles, ADB, AEB, in the same segment, AEB, of a circle, are equal to one another.

Fig. 1.

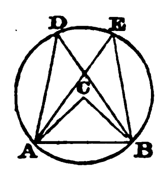
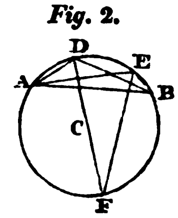


Fig. 2.

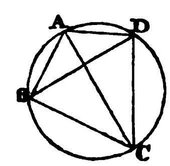


Let C (*fig.* 1) be the centre of the circle; and, first, let the segment AEB be greater than a semi-circle. Draw CA, CB, to the ends of the base of the segment; then each of the angles, ADB, AEB, will be half of the angle ACB (*theorem* 34); therefore the angles ADB and AEB are equal.

Next, let the segment AEB (*fig.* 2,) be less than a semi-circle; draw the diameter DCF, and join EF; and, because the segment ADEF is greater than a semi-circle, by the first case, the angle ADF is equal to AEF. In like manner, because the segment BEDF is greater than a semi-circle, the angle BDF is equal to the angle BEF; therefore the whole angle ADB is equal to the whole angle AEB.

THEOREM 36.

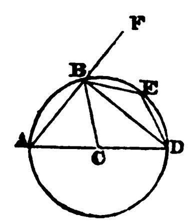
93. The sum of the opposite angles of any quadrilateral, ABCD, inscribed in a circle, is equal to two right angles.



Draw the diagonals AC, BD. In the segment ABCD, the angle ABD is equal to ACD; and, in the segment CBAD, the angle CBD is equal to CAD (*theorem* 35); therefore the whole angle ABC is equal to the sum of the two angles ACD, CAD; and, adding ADC, the sum of the two angles, ABC, ADC, is equal to the sum of the three angles. Now these three angles are the angles of the triangle ADC, and therefore equal to two right angles (*theorem* 27): therefore the sum of the two angles ABC, ADC, is equal to two right angles. In the same manner it may be demonstrated that the sum of the two angles BAD, BCD, is equal to two right angles.

THEOREM 37.

94. An angle ABD, in a semi-circle, is a right angle; an angle BAD, in a segment greater than a semi-circle, is less than a right angle; and an angle, BED, in a segment less than a semi-circle, is greater than a right angle.



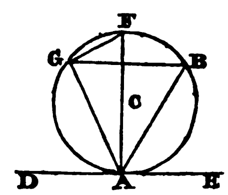
Produce AB to F, draw BC to the centre, and, because CA is equal to CB, the angle CBA is equal to CAB (*theorem* 11). In like manner, because CD is equal to CB, the angle CBD is equal to CDB; therefore the sum of the two angles CBA, CBD, is equal to the sum of the two angles CAB, CDB; that is, the angle ABD is equal to the sum of the two angles CAB, CDB: but this last sum is equal to the angle DBF (*theorem* 27); therefore the angle ABD is equal to the angle DBF: but, when the angles are equal on each side of a

straight line which meets another, each of these angles is a right angle; therefore each of the angles ABD and DBF is a right angle; and, consequently, the angle ABD in a semi-circle is a right angle.

Again, because in the triangle ABD, the angle ABD is a right angle; therefore BAD, which is manifestly in a segment less than a semi-circle, is less than a right angle: and, lastly, because ABED is a quadrilateral in a circle, the sum of the two angles A, E, is equal to two right angles; but the angle A is less than a right angle; therefore E, which is in a segment less than a semi-circle, is greater than a right angle.

THEOREM 38.

95. The angle BAE, contained by a tangent AE to a circle, and a chord AB, drawn from the point of contact, is equal to the angle AGB in the alternate segment.



Let the diameter ACF be drawn, and GF be joined; and, because the angles FGA, FAE, are right-angles (*theorems* 37, 30), and of these FGB, a part of the one, is equal to FAB, a part of the other, (*theorem* 35,) the remainders BAE, BAG, are equal.

ALGEBRA.

96. ALGEBRA is a method of demonstrating propositions, and resolving questions, by means of the letters of the alphabet used as symbols.

LETTERS are employed to denote angles, lines, surfaces, or solids; each being considered as a multitude of units of the kind to which it belongs; and, consequently, as *number*, abstracted from figure. Thus, a letter may denote the number 3, or a line of five equal parts of any measure; as inches, yards, miles, &c. and so on.

Letters which are thus generally employed are called *quantities;* they being the *representation of quantities.*

It frequently happens that a quantity consists of several quantities of the same kind, as of two or more distances to make one distance; these must, therefore, be joined by addition, or by addition and subtraction. In order to indicate this junction, two distinct *signs* will be necessary.

The sign + (*plus*) implies that the quantity which follows it is to be added to that which goes before, and that all the quantities, when more than one, are to be added together into one sum. Thus, $a+b$ shows that b is to be added to a, or that a and b are to be added together.

Again, $a+b+c+d$ implies that b is to be added to a; c, to the sum of a and b; d, to the sum of a, b, c.

The sign — (*minus*) placed between two quantities, denotes that the quantity which follows it is to be subtracted from that which precedes it.

Thus, $m-n$ denotes that the quantity represented by n is to be subtracted from that represented by m. Suppose, for instance, that m is 7, and n 3; then $7-3$ will be 4. Therefore, $m-n$ denotes the remainder arising by subtracting n from m.

Hence Subtraction is an opposite operation to Addition; and, therefore, if any quantity be both added to and subtracted from the same quantity, the quantity thus added and subtracted may be taken away entirely, by which the expression will be in its most simple form: thus, $m+a-a$ is equivalent to m.

When two quantities are equal to each other, this equality is implied by the interposition of the double bar = between each of the quantities. Thus, $m+a-a=m$; as, also, $4+3+6-2=11$. Equal quantities, thus connected, are called *Equations*.

Terms are all those parts of an expression that are separated by the signs of Addition and Subtraction.

Thus, $a+b+c-d$, is a quantity consisting of four *terms*.

Quantities which contain two terms are called *binomials:* thus, $a+b$, or, $a-b$, are binomials.

The Multiplication of two or more factors is indicated by connecting the letters representing the factors; as, ab denotes a product of two factors; abx a

product of the three factors, a, b, and x. Thus, let $a=2$, $b=3$, and $x=5$; then $abx=30$: Again, $mmmm$ signify a product of four factors, of which all are equal: suppose $m=2$, then $mmmm=16$.

When any number of factors are equal to each other, instead of repeating them to that number in the representation of the product, the product will be indicated with less trouble by writing only one of the equal factors and a digit; the latter containing as many units as the factors are in number, over the right-hand side of the factor so written. Thus, instead of aa, bbb, xx, xxx, the same idea will be more conveniently expressed thus, a^2, b^3, x^2, x^3.

The continued product of equal quantities is called a *Power;* the quantity itself is called the *Base* of that power; and the digit, which indicates the number of factors, is called the *Index* or *Exponent* of that power.

When factors of a product consist of compound terms, each compound factor is enclosed within brackets, or parentheses, and the factors thus included are joined to each other by bringing the bracket on the left hand of the one nearly close to that on the right-hand of the other.

Thus, the product of $a+b+c$, $x+a+c+d$, and $a+x$, is represented by $(a+b+c)\,(x+a+c+d)\,(a+x)$. Let $a=1$, $b=2$, $c=3$, $d=4$, and $x=5$, then will $(a+b+c)\,(x+a+c+d)\,(a+x)=(1+2+3)\,(5+1+3+4)\,(1+5)=468$.

Any Power of a compound quantity is represented in a similar manner to that of representing a simple quantity, by inclosing the compound within brackets, and writing the number which indicates the power over the right-hand bracket, and on the right-hand side of that bracket: thus, $(a+b)^3$ denotes the cube of $a+b$, and $(x+y+z)^4$ denotes the fourth power of $x+y+z$.

Division is represented by placing the dividend above the divisor, with a short line between them, as $\frac{a}{b}$; which expression shows how often the quantity a contains the quantity b; or how often the dividend contains the divisor. Let a be 12, and b be 3, then $\frac{a}{b}$ will be 4.

Fractions are represented in the same manner as Division, by placing the numerator above and the denominator below a short line. Thus, $\frac{m}{n}$ indicates a fraction, whose numerator is m, and denominator n. Let m be equal to 2, and n equal to 3; then $\frac{m}{n}$ is equivalent to $\frac{2}{3}$ or two-thirds: or, if we suppose

m to be equal to 17, and n equal to 5, then $\frac{m}{n}$ would be 17-fifths of unity, or by dividing the numerator m, which is equivalent to 17, by the denominator n, which is equivalent to 5: the quotient will be 3 and $\frac{2}{5}$.

All expressions of quantity are said to be *Simple* when the operations are indicated by one or more letters, either in Multiplication or Division, without the intervention of the signs + or −, as in the following: $a, ab, \frac{a}{b}, \frac{ab}{c}$; which are all *Simple* expressions.

Known quantities are generally represented by the initial letters, a, b, c, &c. of the alphabet, or by numbers; and the *unknown* quantities by the final letters, v, w, x, y, z.

A Co-efficient is the number prefixed to any quantity. Thus, in the expression $5x$, the number 5 is the co-efficient of x; or, if x represent a quantity to be discovered by an operation, and a a quantity already known, then, in the expression ax, the quantity a is called the co-efficient of x.

Having explained the forms which indicate the operations of Simple Quantities, we shall now explain the rules for those performed upon Compounds.

ADDITION OF ALGEBRA.

97. To add any number of simple affirmative quantities, which are of the same kind, together, or any number of quantities that have a common factor:

Prefix the sum of the co-efficients to the quantity, and the product will represent the sum; observing that, when no co-efficient is written, the co-efficient is understood to be unity: and, when the co-efficients are expressed by letters, these letters are to be joined with the sign + within brackets, and the common quantity adjoined or subjoined.

In the following examples let the sum be put equal to S.

Example 1.—Add a, a, a, a, a, together; then $5a = S$.

Example 2.—Add ax, $2\,ax$, $3\,ax$, together; then $6ax = S$.

Ex. 3.—Add ax, bx, cx, dx, ex, fx, together; then $(a+b+c+d+e+f)x = S$.

98. To add any number of simple affirmative quantities of different kinds together:

Connect the whole to be added by the sign + ; and, if two or more of these quantities are to be found, of the same kind, they must be united into one simple quantity, or term, as above.

Example 1.—Add a, b, c, d, together; then $a+b+c+d=$ S.

Example 2.—Add a, b, b, c, b, dx, ey, together; $a+3b+c+dx+ey=$ S.

99. To add quantities together which have different signs:

Join all the quantities into one expression for the sum; observing to prefix the same sign to each quantity that it had before the whole were united.

Example.—Add $a, -bx, cd, -2bx$, together; then, $a+cd-3bx=$ S.

100. To add Compound Quantities together:

Connect all the quantities, in the several parts, to be added into one expression, giving each quantity the same sign that it had before, in each separate part, and observing to unite such terms as may be found of the same kind.

Example.—Add $5bx+\frac{4c}{2}$, $5ab-\frac{bc}{e}$, $2bx-3ab+3ex$, together. The answer will be $7bx+2ab+\frac{4c}{2}-\frac{bc}{e}+3ex=$ S.

SUBTRACTION OF ALGEBRA.

101. To subtract one simple quantity from another:

Join the quantity to be subtracted to that from which the subtraction is to be made, with a different sign to the original one. Let the difference be put equal to D.

Example 1.—Subtract n from m; then $m-n=$ D.

Example 2.—Subtract $3ab$ from $7ab$; then $7ab-3ab=4ab=$ D.

102. To subtract a compound quantity either from a simple or compound quantity.

Subjoin the terms of the quantity to be taken away, with their signs changed, to that from which they are to be taken; observing that, when two terms are of the same kind, they must be united.

Ex.—From $xy+4b-3c$, subtract $bx-5b+4c$; then $xy+9b-7c-bx=$ D.

It is evident that changing the signs of the terms of the quantity to be taken away cannot affect its aggregate or value, considered independently of

its signs. For, if they are of different kinds, the difference must be the same after the change as before it took place.

Thus, let $5-2+7-3$, be a quantity to be subtracted.

$$\text{Then} \quad 5-2+7-3=7$$
$$\text{and} \quad -5+2-7+3=-7$$

Hence we see the reason for changing the signs of the quantity to be subtracted.

MULTIPLICATION OF ALGEBRA.

103. To find the algebraic product of two compound factors, or of one simple and the other compound.

If one of the factors be a simple quantity, let that factor be made the multiplier; then join the multiplier to every term of the multiplicand, and prefix the sign + to each product when its factors have like signs; but prefix the sign − when its factors have unlike signs; then the sum of all the products is the total product.

If the multiplier consist of more than one term, proceed with every term of the multiplier in the same manner as if it had but one term; then the sum of all the simple products is the whole product; observing that, all such simple products, as have a common factor, may be united together.

Example 1.—Multiply $a+b$ by $a+b$.

Operation...
$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\quad +ab+b^2} \\ a^2+2ab+b^2=(a+b)^2 \end{array}$$

104. So that the square of any binomial consists of the square of each part and twice their product.

Hence, if we see such a quantity as $x^2+2ax+a^2$, we shall immediately know that it is the square of the binomial $x+a$.

Example 2.—Multiply $a-b$ by $a-b$, or find the square of $a-b$.

Operation. . .
$$\begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ \underline{\quad -ab+b^2} \\ a^2-2ab+b^2=(a-b)^2 \end{array}$$

105. Hence the square of any binomial, which has one of its parts negative, is the same which ever of the parts be negative; for the square of each of the parts is always affirmative, and twice the product of the two parts is negative; so that the square of $a-b$ is the very same as the square of $b-a$.

Example 3.—Multiply $a+b$ by $a-b$.

Operation. . .
$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad -ab-b^2} \\ a^2 \quad * \ -b^2=(a+b)\,(a-b) \end{array}$$

106. Hence we shall always know, by bare inspection only, that the difference of two squares is the product of the sum and difference of the roots: hence $x^2-y^2=(x+y)\,(x-y)$, and, reciprocally, that the product of the sum and difference of any two quantities is equal to the difference of their squares.

Thus $(a+x)\,(a-x)=a^2-x^2$.

ALGEBRAIC DIVISION AND FRACTIONS.

107. Division is the converse of Multiplication; therefore, if the signs be alike in the divisor and dividend, the quotient will be affirmative; but if unlike, the quotient will be negative. The general rule is to place the dividend in the form of a numerator, and the divisor in that of a denominator; expunge like quantities from both, and divide the co-efficients by the greatest common measure.

Example 1.—Divide $3ab$ by b........ $\frac{3ab}{b}=3a$

Example 2.—Divide $-abc$ by $-bc$ .. $\frac{-abc}{-3bc}=\frac{a}{3}$

Example 3.—Divide $4ac$ by $16ba$ .... $\frac{4ac}{16ba}=\frac{c}{4b}$

108. Powers of the same root are divided by subtracting their exponents.

Example 1.—Divide b^3 by b^2. .. .. $\frac{b^3}{b^2}=b$..

Example 2.—Divide $a^4b^3c^2$ by a^2b^2.... $\frac{a^4b^3c^2}{a^2b^2}=a^2bc^2$.

109. A FRACTION is multiplied by any quantity by joining it to the numerator; thus, the product of r and $\frac{m}{n}$ is represented by $\frac{rm}{n}$, which is the product of rm divided by n, or the number of times that the product rm contains n.

Since the operation of division is opposite to that of multiplication, if a fraction be multiplied by a quantity equal to its denominator, both the denominator and the multiplier may be taken away from the result; thus, if $\frac{a}{b}$ be multiplied by b, the result is $\frac{ab}{b}$, which is equivalent to the quantity a only.

110. The *Terms* of a fraction are its numerator and denominator.

If the terms of a fraction be equally multiplied, that is, multiplied by the same quantity, the value of that fraction will be the same as before; thus $\frac{a}{b}=\frac{ma}{mb}$: and, if the terms of a fraction be equally divided by the same quantity, the result will be equal to the original quantity.

ALGEBRAIC EQUATIONS.

111. The *resolution of an equation* is the mode of finding the value of the unknown quantity, in terms of those which are given.

The principles of this resolution depend on the following AXIOMS, which are similar to those already given at the beginning of the Geometry, in page 15.

1. If to each side of an equation the same quantity be added, the sums will be an equation still.

2. If from each side of an equation the same quantity be subtracted, the remainders will still be an equation.

3. If each side of an equation be multiplied by the same quantity, the products will be an equation.

4. If each side of an equation be divided by the same quantity, the quotients will still be an equation.

OF PROPORTION.

112. Definition.—Four quantities are proportionals when the first contains some part of the second, as often as the third contains the like part of the fourth.

THEOREM 39.

113. If four quantities, a, b, c, d, are proportionals, the product of the two extremes will be equal to the product of the two means.

Let the first, a, contain the nth part of the second b, m times; then, by the definition, the third, c, will contain the nth part of d also m times; now the nth part of b is $\frac{b}{n}$, and the nth part of d is $\frac{d}{n}$; therefore $\frac{a}{\frac{b}{n}}=m$; and $\frac{c}{\frac{d}{n}}=m$; wherefore $\frac{a}{b}=\frac{c}{d}$.

Multiply each side of this equation by bd, and $\frac{abd}{b}=\frac{bcd}{d}$; therefore, dividing the terms of the fraction on the first side by b, and the terms of the fraction on the second side by d, which are common, $ad=bc$.

THEOREM 40.

114. If any equation consist of the product of two quantities on each side, and if the four factors be placed in a row, so that the two factors on either side may occupy the middle place of the four, and the other two each one of the extreme places; the four factors, thus taken in order, will be proportionals.

Let $ad=bc$: Now here are four different ways of taking out the quantities; but in which ever of these ways they are taken, we shall always have $ad=bc$, by multiplying the two extreme terms together, and the two middle terms together.

Let, therefore, a, b, c, d, be one of the four; then, since $ad=bc$, divide both sides by bd and $\frac{ad}{bd}=\frac{bc}{bd}$; that is, $\frac{a}{b}=\frac{c}{d}$; now let h be the common measure of a

and b, supposing h contained in a, m times, and in b, n times; then $\frac{a}{b}=\frac{mh}{nh}=\frac{m}{n}$; therefore $\frac{a}{\frac{b}{n}}=m$; but $\frac{b}{n}$ is the nth part of b; therefore the nth part of b is contained in a, m times.

Again, let $c=mi$, and we shall have $\frac{mh}{nh}=\frac{mi}{d}$; wherefore, by this equation, $d=ni$; therefore $\frac{c}{d}=\frac{mi}{ni}=\frac{m}{n}$; and, consequently, $\frac{c}{\frac{d}{n}}=m$: wherefore, also, c contains the nth part of d, m times.

115. Corollary.—Hence, if two fractions are equal, the numerator of the one will be to its denominator, as the numerator of the other is to its denominator.

To indicate that four quantities, a, b, c, d, are proportionals, the sign : is placed between the first two and between the last two; and the sign :: is placed between the two terms that stand in the middle; thus, $a : b :: c : d$, is read, as a is to b, so is c to d.

116. Four proportionals are termed a *proportion*.

Of four proportional quantities, the last term is called the fourth proportional to the other three.

The first and third terms of a proportion are called the *antecedents*, and the second and fourth terms the *consequents*.

SCHOLIUM.

117. Since, from $ad=bc$, we may choose four different ways of taking out the first term, in order to make a proportion; and since there are two ways of making choice of the second term after the first, there are eight ways in which the positions of the terms will be different: these are—

$$a : b :: c : d$$
$$a : c :: b : d$$
$$b : a :: d : c$$
$$b : d :: a : c$$
$$c : a :: d : b$$
$$c : d :: a : b$$
$$d : b :: c : a$$
$$d : c :: b : a$$

118. So that, in every two sets of proportionals, where the same quantity stands first, the terms of the one set are placed alternately to those of the other; and we may also observe that, four of these sets of proportionals have their terms inverted with regard to the other four, and are therefore the same when read in contrary order.

THEOREM 41.

119. If the corresponding terms of any number of proportions are multiplied together, the products, taken in the same order, will be proportionals.

$$\begin{aligned} \text{Thus, as } a &: b :: c : d \\ e &: f :: g : h \\ i &: k :: l : m \\ \text{then, as } aei &: bfk :: cgl : dhm \end{aligned}$$

$$\begin{aligned} \text{For } ad &= bc \\ eh &= fg \\ im &= kl \\ \text{therefore, } adehim &= bcfgkl. \end{aligned}$$

Consequently, $aei : bfk :: cgl : dhm$; therefore the proposition is manifest.

120. Corollary.—Hence, if the proportions are the same as the first, we shall have $a^3 : b^3 :: c^3 : d^3$.

121. The proportion which is formed by the multiplication of the corresponding terms of two or more proportions is said to be compounded of these proportions.

When two of the proportions to be compounded are the same, the proportion compounded of the two is said to be the duplicate of either: thus, $a^2 : b^2 :: c^2 : d^2$ is in duplicate proportion to that of $a : b :: c : d$.

When three proportionals, which are the same, are compounded, the compound is said to be triplicate to either of the simple ones: thus, $a^3 : b^3 :: c^3 : d^3$, is the triplicate proportion of $a : b :: c : d$.

And so on to the succeeding orders.

THEOREM 42.

122. If four quantities, a, b, c, d, be proportionals, the first will be to the sum of the first and second as the third is to the sum of the third and fourth.

For, since a, b, c, d, are proportionals, $ad=bc$. To each side of this equation add the product ac, and we have $ad+ac=bc+ac$; that is, $a(c+d)=c(a+b)$.

$$\text{Therefore, } a : a+b :: c : c+d.$$

123. Corollary.—Since out of the two equal products ad, bc, four combinations in twos may be chosen, viz.

$$ab,\ ac,\ bd,\ cd,$$

we may therefore have the four following equations all different, by adding each combination to each side of the equation $ad=bc$, viz.

$$\begin{aligned}
\text{No. } 1 \ldots\ldots a(c+d) &= c(a+b)\\
2 \ldots\ldots a(b+d) &= b(a+c)\\
3 \ldots\ldots d(a+b) &= b(c+d)\\
4 \ldots\ldots d(a+c) &= c(b+d)
\end{aligned}$$

Since each of these equations will give eight sets of proportionals; therefore the whole four will give thirty-two.

THEOREM 43.

124. If four quantities, a, b, c, d, be proportionals, as the first is to the difference of the first and second, so is the third to the difference of the third and fourth.

For, since a, b, c, d, are proportionals, $ad=bc$. Subtract each side of this equation from the product ac, and we have $ac-ad=ac-bc$; that is, $a(c-d)=c(a-b)$, whence $a : a-b :: c : c-d$.

125. Corollary.—Since out of the two equal products ad, bc, we may choose the four combinations in twos, viz. ab, ac, bd, cd; and may therefore have the four following equations, by subtracting each side of the original equation, $ad=bc$, from each combination.

$$\text{No. }1\ldots a(c-d)=c(a-b)$$
$$2\ldots a(b-d)=b(a-c)$$
$$3\ldots d(b-a)=b(d-c)$$
$$4\ldots d(c-a)=c(d-b)$$

126. Again, by subtracting each of the products, ab, ac, bd, cd, from each side of the original equation, $ad=bc$, we have—

$$\text{No. }1\ldots a(d-c)=c(b-a)$$
$$2\ldots a(d-b)=b(c-a)$$
$$3\ldots d(a-b)=b(c-d)$$
$$4\ldots d(a-c)=c(b-d)$$

Now, since every one of these eight different equations will give eight sets of proportionals, in each set of which the situations of the terms will be varied; therefore the whole will give sixty-four sets of proportionals, in which the terms will have different situations in every two sets.

THEOREM 44.

127. If four quantities, a, b, c, d, be proportionals, it will be, as the sum of the first and second is to their difference, so is the sum of the third and fourth to their difference.

For, dividing the equation, No. 3, pr. 123, by the equation, No. 3, pr. 126, we shall have—

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}$$

And, by dividing the equation, No. 3, in 123, by the equation, No. 3, in pr. 125, we have,

$$\frac{a+b}{b-a}=\frac{c+d}{d-c}$$

Wherefore $a+b : a\sim b :: c+d : c\sim d$*

128. Three quantities are said to be proportionals when the first is to the middle quantity as the middle quantity is to the third; thus, let a, b, c, be three proportionals; then $a : b :: b : c$.

* The character $\sim$ between two quantities signifies the difference, as $5\sim 9=4$.

THEOREM 45.

129. If three quantities be proportionals, the product of the two extremes is equal to the square of the mean.

Let $a, b, c,$ be the three proportionals; then, $ac=b^2$, for, since $\frac{b}{a}=\frac{c}{b}$, multiply both sides of this equation by ab, and $\frac{ab^2}{a}=\frac{abc}{b}$, that is, $b^2=ac$.

THEOREM 46.

130. If the first and second terms of two proportions are alike, the third and fourth terms of both, placed in the order of their antecedents and consequents, will be proportionals.

Let $a : b :: c : d$, and $a : b :: e : f$; then will $c : d :: e : f$. From the first proportion we have $ad = bc$, and from the second we have $be = af$; therefore, multiplying the corresponding sides of these equations, we have $adbe=bcaf$; and, throwing out the common factors on each side of this equation, there will remain $de=cf$; wherefore, $c : d :: e : f$.

THEOREM 47.

131. In any number of proportionals, of which all the ratios are equal, it will be, as the antecedent of any ratio is to its consequent, so is the sum of all the antecedents of the other ratios to the sum of all the consequents.

For, let $\frac{a}{b}=\frac{c}{d}$, $\frac{c}{d}=\frac{e}{f}$, $\frac{e}{f}=\frac{g}{h}$, then will $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$.

$$\text{Therefore, } ad = bc$$
$$af = be$$
$$ah = bg$$
$$\text{Whence } a(d+f+h) = b(c+e+g.)$$

Wherefore, $a : b :: c+e+g : d+f+h$, as was to be shown.

GEOMETRY, CONTINUED.

HAVING now explained so much of the principles of Algebra and Proportion as may be requisite to elucidate the subsequent propositions, we again proceed with the ELEMENTS OF GEOMETRY. The geometric definitions, &c. have been given, generally, in pages 10 to 14; but to those already explained are to be added several which follow, as it now becomes necessary that they, also, should be known.

132. EQUIVALENT FIGURES are such as have equal surfaces, without regard to their form.

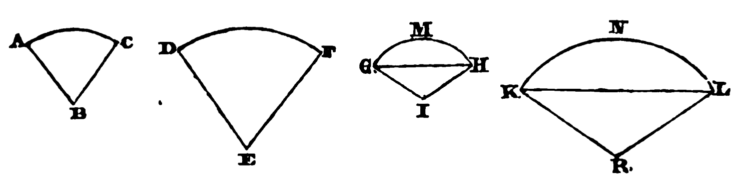
133. IDENTICAL FIGURES are such as would entirely coincide, if the one be applied to the other.

134. In EQUIANGULAR FIGURES, the sides which contain the equal angles, and which adjoin equal angles, are *homologous*.

135. Two figures are *similar*, when the angles of the one are equal to the angles of the other, each to each, and the homologous sides are proportionals.

136. In two CIRCLES, similar sectors, similar arcs, or similar segments, are those which have equal angles at the centre.

Thus, if the sector ABC be similar to the sector DEF, then the angle ABC will be



equal to the angle DEF; or, if the arc AC be similar to the arc DF, then the angle at B will be equal to the angle at E. Also, if the segment GMH be similar to the segment KNL, the angle I will be equal to the angle R.

137. The AREA of a figure is the quantity of surface, containing a certain number of units of any given scale; as of inches, feet, yards, &c.

THEOREM 48.

138. Parallelograms which have equal bases and equal altitudes are equal.

Fig. 1.

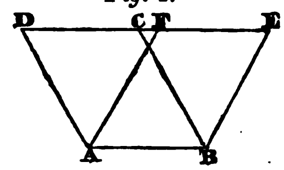
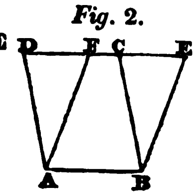


Fig. 2.



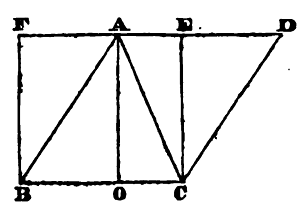
Take the two parallelograms, ABCD, and ABEF, upon the same base, AB, and between the same parallels, AB and DE; these parallelograms are equal.

For, in the parallelogram ABCD (*fig.* 1), the opposite sides CD and AB are equal; and in the parallelogram ABEF, the opposite sides EF and AB are equal; therefore EF is equal to CD (84)*. Again, in the parallelogram ABCD (*fig.* 2), the side CD is equal to AB; and, in the parallelogram ABEF, the side FE is equal to AB; therefore EF is equal to CD. Now, in *fig.* 1, since CD is equal to EF, add CF to both; then will DF be equal to CE. In *fig.* 2, take away the common part CF, and there will remain DF equal to CE. Therefore, in each of these figures, the three straight lines AD, DF, FA, are respectively equal to the three straight lines BC, CE, EB; and, consequently, the triangle ADF is equal to the triangle BCE: therefore, from the quadrilateral ABED take away the triangle BCE, there will remain the parallelogram ABCD; and, from the same quadrilateral, take away the equal triangle ADF, and there will remain the parallelogram ABEF: therefore the parallelogram ABCD is equal to the parallelogram ABEF.

139. Corollary.—Every parallelogram is equal to a rectangle, of the same base and altitude.

THEOREM 49.

140. Any triangle is equal to half a parallelogram of the same base and altitude.



For the triangle ABC is equal to the triangle ACD, and the parallelogram ABCD is equal to the sum of both triangles; and, consequently, double to one of them.

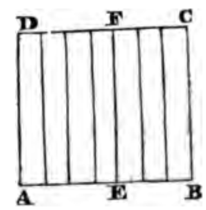
* The figures thus inserted in a parenthesis refer to a preceding or a following paragraph; as, in this instance, to 84, on page 29.

141. **Corollary 1.**—Hence every triangle is half a rectangle, having the same base and altitude.

142. **Corollary 2.**—Triangles which have equal bases and equal altitudes are equal.

THEOREM 50.

143. Rectangles, of the same altitude, are to one another as their bases.

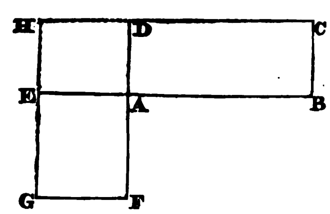


Let ABCD, AEFD, be two rectangles, which have a common altitude AD; they are to one another as their bases AB, AE.

For, suppose that the base AB contains seven equal parts, and that the base AE contains four similar parts; then, if AB be divided into seven equal parts, AE will contain four of them. At each point of division draw a perpendicular to the base, and these will divide the figure ABCD into seven equal rectangles (138); and, as AB contains seven such parts as AE contains four, the rectangle ABCD will also contain seven such parts as the rectangle AEFD contains four; therefore the bases AB, AE, have the same ratio that the rectangles ABCD, AEFG, have.

THEOREM 51.

144. Rectangles are to one another as the products of the numbers which express their bases and altitudes.



Let ABCD, AEGF, be two rectangles, and let some line taken, as a unit, be contained m times in AB, the base of the one, and n times in AD, its altitude; also p times in AE, the base of the other, and q times in AF, its altitude; the rectangle ABCD shall be to the rectangle AEGF, as the product mn is to the product pq.

Let the rectangles be so placed that their bases AB, AE, may be in a straight line; then their altitudes AD, AF, shall also form a straight line (48). Complete the rectangle EADH; and, because this rectangle has the same alti-

tude as the rectangle ABCD, when EA and AB are taken as their bases; and the same altitude as the rectangle AEGF, when AD, AF, are taken as their bases; we have the rectangle ABCD : ADHE :: AB : AE :: $m : p$....(143)

But........ $m : p :: mn : pn$

therefore, ABCD : ADHE :: $mn : pn$.

In like manner, ADHE : AEGF :: $pn : pq$

But, placing the terms of these two sets of proportionals alternately, we have,

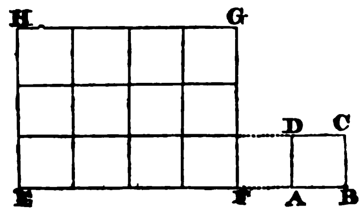
ADHE : pn :: ABCD : mn

and.............. ADHE : pn :: AEGF : pq

therefore, by equality, ABCD : mn :: AEGF : pq

therefore, alternately, ABCD : AEGF :: $mn : pq$.

145. Observation.—If ABCD, one of the rectangles, be a square, having the measuring unit for its side; this square may be taken as the measuring unit of its surfaces; because the linear unit, AB, is contained p times in EF, and q times in EH, by the proposition.



$$1 \times 1 : pq :: ABCD : EFGH.$$

Hence the rectangle EFGH will contain the superficial unit ABCD, as often as the numeral product pq contains unity.

Consequently, the product pq will express the area of the rectangle, or will indicate how often it contains the unit of its surfaces.

Thus, if EF contains the linear unit AB four times, and EH contains it three times, the area EFGH will be $3 \times 4 = 12$: that is, equal to twelve times a square whose side AB is $= 1$.

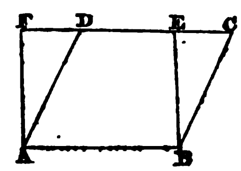
In consequence of the surface of the rectangle EFGH being expressed by the product of its sides, the rectangle, or its area, may be denoted by the symbol EF $\times$ FG, in conformity to the manner of expressing a product in arithmetic.

However, instead of expressing the area of a square, made on a line AB, thus, AB $\times$ AB; it is thus expressed AB^2.

146. NOTE.—A rectangle is said to be contained by two of its sides, about any one of its angles.

THEOREM 52.

147. The area of a parallelogram is equal to the product of its base and altitude.



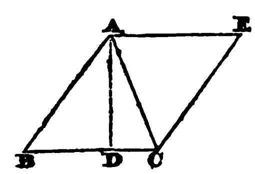
For the parallelogram ABCD is equal to the rectangle ABEF, which has the same base AB, and the same altitude (138), and this last is measured by AB × BE, or by AB × AF; that is the product of the base of the parallelogram and its altitude.

148. COROLLARY.—Parallelograms of the same base are to one another as their altitudes; and parallelograms of the same altitudes are to one another as their bases.

For, in the former case, put B for their common base, and A, a, for their altitudes; then we have $B \times A : B \times a :: A : a$. And, in the latter case, put A for their common altitude, and B, b, for their bases; then $B \times A : b \times A :: B : b$.

THEOREM 53.

149. The area of a triangle is equal to the product of the base by half its altitude.

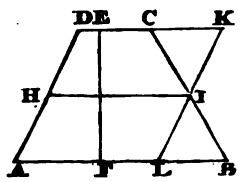


For the triangle ABC is half the parallelogram ABCE, which has the same base, BC, and the same altitude AD (140); but the area of the parallelogram is BC × AD (147), therefore, the area of the triangle is $\frac{1}{2}$BC × AD, or BC × $\frac{1}{2}$AD.

150. COROLLARY.—Two triangles of the same base are to one another as their altitudes; and two triangles, of the same altitude, are to one another as their bases.

THEOREM 54.

151. The area of every trapezoid, ABCD, is equal to the product of half the sum of its parallel sides, AB, DC, by its altitude, EF.



Through I, the middle of the side BC, draw KL, parallel to the opposite side AD, and produce DC until it meet KL in K. In the triangles IBL, ICK, the side IB is equal to IC, and the angle B equal to C (74), the angle BIL equal to CIK; therefore the triangles are equal (54), and the side CK equal to BL. Now the parallelogram ALKD is the sum of the polygon ALICD; and the triangle CIK, and the trapezoid ABCD, is the sum of the same polygon and the triangle BIL; therefore, the trapezoid ABCD is equal to the parallelogram ALKD, and has, for its measure, $AL \times EF$. Now AL is equal to DK, and BL equal to CK; and $CD = DK - CK$; but DK is equal to AL, and CK equal to BL;

Therefore, $CD = AL - BL$.

But...... $AB = AL + BL$.

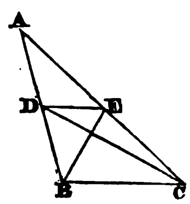
Therefore, $AB + CD = 2AL$

Consequently $\frac{1}{2}(AB + CD) = AL$.

It follows that .. $AL \times EF = \frac{1}{2}(AB + CD) \times EF$.

THEOREM 55.

152. A straight line, DE, drawn parallel to the side of a triangle ABC, divides the other sides AB, AC, proportionally, or so that AD : DB :: AE : EC.



Join BE and DC: the two triangles BDE, CDE, have the same base, DE, and they have also the same altitude, because BC is parallel to DE; and, consequently, the triangles DBE and DCE are equal. Since the triangles BED and AED have the same altitude, they are to one another as their bases; and since the triangles CED and AED have the same altitude, they are to one another as their bases.

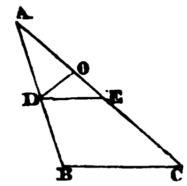
Therefore, the triangle BDE : ADE :: BD : DA. But since the triangle BDE is equal to the triangle CED, therefore the triangle CED : ADE :: BD : DA.

But the triangle CED : ADE :: CE : EA.

It follows that, BD : DA :: CE : EA.

THEOREM 56.

153. If the two sides, AB, AC, of a triangle be cut proportionally by the line DE, so that AD : DB :: AE : EC, the line DE shall be parallel to the remaining side of the triangle.



For, if DE be not parallel to BC, some other line, DO, will be parallel to BC; then, by the preceding proposition,

AD : DB :: AO : OC.

And, by hypothesis, AD : DB :: AE : EC.

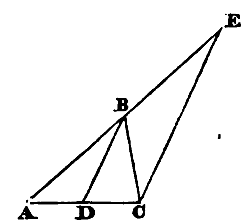
Therefore, ... AO : OC :: AE : EC.

And, by addition, AC : OC :: AC : EC.

And hence OC must be equal to EC; which is impossible, unless the point O coincide with E; therefore no line besides DE can be parallel to BC.

THEOREM 57.

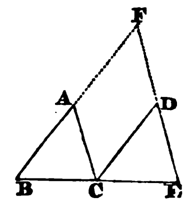
154. A line, BD, which bisects any angle, ABC, of a triangle, will divide the opposite side AC into two segments, AD, DC, which shall have the same ratio as the other two sides, AB, BC, of the triangle.



From C, one extremity of the base, draw CE, parallel to BD, meeting AB produced in E. Then the angle ABD is equal to the angle BEC (75), and the angle CBD equal to BCE (74); but, by hypothesis, the angle ABD is equal to CBD; therefore the angle BEC is equal to BCE: hence the side BC is equal to BE (63). Now, because ACE is a triangle, and BD is drawn parallel to one of its sides, AD : DC :: AB : BE (152); but, since BE is equal to BC; therefore AD : DC :: AB : BC.

THEOREM 58.

155. Two equiangular triangles have their sides proportional, and are similar to each other.



Let ABC, DCE, be two triangles, which have their angles equal, each to each; viz. BAC equal to CDE, ABC equal to DCE, and ACB equal to DEC; the homologous sides, or the sides adjacent to the equal angles, shall be proportionals; that is, BC : CE :: BA : CD, and BA : CD :: AC : DE.

Place the homologous sides BC, CE, in a straight line; and, because the angles B and E are together less than two right angles, the lines BA and ED shall meet, if produced (76): let them meet in F. Then, since BCE is a straight line, and the angle BCA equal to E, AC is parallel to EF. In like manner, because the angle DCE is equal to B, the straight line CD is parallel to BF; therefore ACDF is a parallelogram.

156. In the triangle BFE, the straight line AC is parallel to FE; wherefore BC : CE :: BA : AF (152). Again, in the same triangle, BFE, CD is parallel to BF; therefore, BC : CE :: FD : DE; but, by substituting CD for its equal AF, in the first set of proportionals, and AC for its equal FD in the second set,

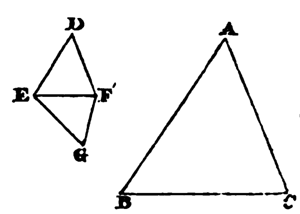
we have............ BC : CE :: BA : CD by the first,
and .... BC : CE :: AC : DE by the second,
therefore, by equality, BA : CD :: AC : DE;

therefore the homologous sides are proportionals; and, because the triangles are equiangular, they are similar.

SCHOLIUM.—It may be remarked that the homologous sides are opposite to the equal angles.

THEOREM 59.

157. Two triangles, which have their homologous sides proportionals, are equiangular and similar.



Suppose that BC : EF :: AB : DE
and that.... AB : DE :: AC : DF

the triangles ABC, DEF, have their angles equal: viz. A equal to D, B equal to E, and C equal to F. At the point E make the angle FEG equal to B, and at the

point F make the angle EFG equal to C, then G shall be equal to A (80), and the triangles GEF, ABC, shall be equiangular; therefore,

by the preceding prop. BC : EF :: AB : EG

and, by hypothesis, .. BC : EF :: AB : DE; therefore EG is equal to DE.

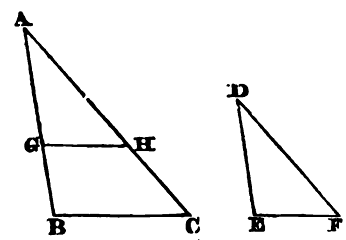
In like manner....... BC : EF :: AC : FG

and, by hypothesis, .. BC : EF :: AC : DF; therefore FG is equal to DF.

Thus, it appears that, the triangles DEF, GEF, have their three sides equal, each to each, therefore they are equal (58). But, by construction, the triangle GEF is equiangular to the triangle ABC; therefore, also, the triangles DEF, ABC, are equiangular and similar.

THEOREM 60.

158. Two triangles which have an angle of the one equal to an angle of the other, and the sides about them proportionals, are similar.



Let the angle A equal D, and suppose that AB : DE :: AC : DF, the triangle ABC is similar to DEF.

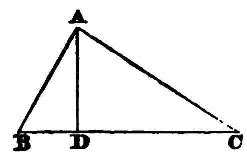
Take AG equal to DE, and draw GH parallel to BC, the angle AGH shall be equal to ABC (75), and the triangle AGH equiangular to the triangle ABC; therefore AB : AG :: AC : AH; but AG is equal to DE;

therefore........ AB : DE :: AC : AH,

but, by hypothesis, AB : DE :: AC : DF; therefore AH is equal to DF. The two triangles AGH, DEF, have therefore an angle of the one equal to an angle of the other, and the sides containing these angles equal; therefore they are equal (53); but the triangle AGH is similar to ABC.

THEOREM 61.

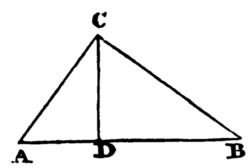
159. A perpendicular, AD, drawn from the right angle, A, of a right-angled triangle, upon the hypothenuse, or longest side, BC, will divide that triangle into two others, which will be similar to each other, and to the whole.



The triangles BAD and BAC have the common angle B; and, besides, the right angle BDA is equal to the right angle BAC; therefore, the third angle BAD of the one is equal to the third angle C of the other (80); therefore the two triangles are equiangular and similar. In like manner it may be demonstrated that the triangle DAC is equiangular and similar to the triangle BAC; therefore the three triangles are equiangular and similar to one another.

THEOREM 62.

160. The square described upon the hypothenuse, or longest side, is equal to the squares described upon the other two sides.

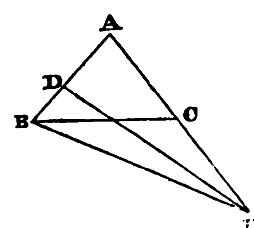


From the right angle C draw CD, perpendicular to the hypothenuse AB; then the triangle ABC is divided into two triangles, ADC, CDB, which are similar to one another, and to the whole triangle ABC (159);

therefore, by the similar triangles, ABC, CBD, . . $AB : BC :: BC : BD$;
again, by the similar triangles, BAC, CAD . . . . $AB : AC :: AC : AD$;
therefore, reducing the first to an equation, . . . $AB \times BD = BC^2$
and, reducing the second analogy to an equation, $AB \times AD = AC^2$
then, adding these two equations . . . . . . $AB \times (AD + BD) = AC^2 + BC^2$
but, since AB is equal to the sum of the two lines AD, DB, therefore $AB^2 = AC^2 + BC^2$,

THEOREM 63.

161. Two triangles, which have an angle of the one equal to an angle of the other, are to each other as the rectangle of the sides about the equal angles.



Suppose* the two triangles joined, so as to have a common angle, and let the two triangles be ABC, ADE. Draw the straight line BE.

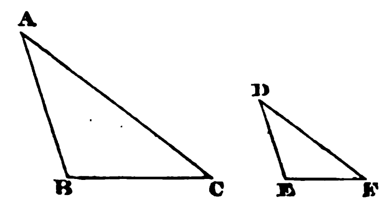
Now the triangle ABE : tria. ADE :: AB : AD;

Therefore the triangle ABE : tria. ADE :: AB × AE : AD × AE.

Or, alternately, the triangle ABE : AB × AE :: the triangle ADE : AD × AE.
In like manner, the triangle ABE : AB × AE :: the triangle ABC : AB × AC.
Therefore, by equality, the tria. ABC : tria. ADE :: AB × AC : AD × AE.

THEOREM 64.

162. Similar triangles are to one another as the squares of their homologous sides.



Let the angle A be equal to the angle D, and the angle B equal to E.

Then AB : DE :: AC : DF (155)

and AB : DE :: AB : DE.

therefore, by multiplying the corresponding terms, we have $AB^2 : DE^2$ :: AC × AB : DF × DE.

But the triangle BAC : triangle EDF :: AC × AB : DF × DE (162).

Therefore the triangle ABC : triangle DEF :: $AB^2 : DE^2$.

Or thus, let △ signify a triangle; then (162)—

△ABC : △DEF :: AB × AC : DE × DF.

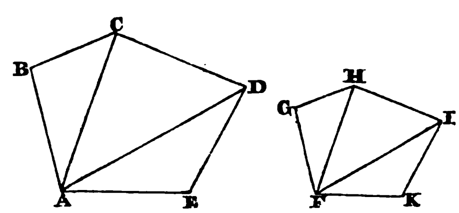
△ABC : △DEF :: AB × BC : DE × EF.

AC × BC : DF × EF :: △ABC : △DEF.

Therefore, by multiplication, △ABC : △DEF :: AB^2 : DE^2.

THEOREM 65.

163. Similar polygons are composed of the same number of triangles, which are similar, and similarly situated.



In the polygon ABCDE, draw from any angle, A, the diagonals AC, AD; and, in the other polygon, FGHIK, draw, in like manner, from the angle F, which is homologous to A, the diagonals FH, FI.

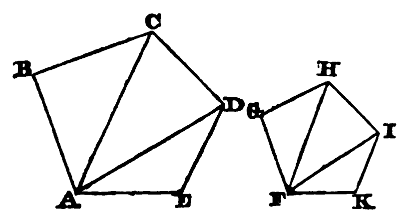
Since the polygons are similar, the angle B is equal to its homologous angle G; and, besides, AB : BC :: FG : GH; therefore, the triangles ABC and FGH

are similar (158), and the angle BCA is equal to GHF; these equal angles being taken from the equal angles BCD, GHI, the remainders ACD, FHI, are equal: but, since the triangles ABC and FGH are similar, we have AC : FH :: BC : GH; and, because of the similitude of the polygons, we have BC : GH :: CD : HI; therefore, AC : FH :: CD : HI. Now it has been shown that the angle ACD is equal to FHI; therefore the triangles ACD, FHI, are similar (158).

In like manner, it may be demonstrated that, the remaining triangles of the two polygons are similar; therefore the polygons are composed of the same number of similar triangles, similarly situated.

THEOREM 66.

164. The perimeters of similar polygons are to one another as their homologous sides.



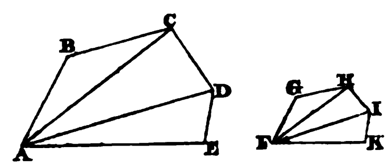
AB : FG :: BC : GH.

BC : GH :: CD : HI.

Therefore, AB : FG :: AB + BC + CD :: FG + GH + HI (131); wherefore AB is to FG as the perimeter of the polygon ABCD is to the perimeter of the polygon FGHIK.

THEOREM 67.

165. The areas of similar polygons are as the squares of their homologous sides.



Let the polygons be ABCDE and FGHIK; from any angle, A, draw the diagonals AC, AD; and, from the homologous angle F, draw the diagonals FH, FI; then the triangles ABC, ACD, ADE, are respectively equal, and similar to the triangles FGH, FHI, FIK.

Therefore the triangle ABC : triangle FGH :: AC^2 : FH^2

And .... the triangle ACD : triangle FHI :: AC^2 : FH^2.

Therefore the triangle ABC : triangle FGH :: ACD : FHI.

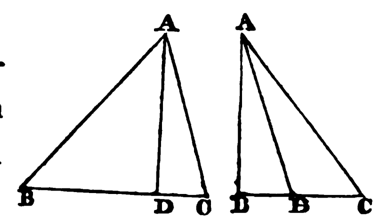
In the same manner it may be demonstrated that the triangle ACD : triangle FHI :: ADE : FIK, and so on, if the polygons consist of more triangles.

Hence (131) the triangle ABC is to the triangle FGH as the sum of the triangles ABC, ACD, ADE, to the sum of the triangles FGH, FHI, FIK; but the sum of the triangles ABC, ACD, ADE, compose the whole polygon ABCDE, and the sum of the triangles FGH, FHI, FIK, compose the polygon FGHIK; wherefore the triangle ABC is to the triangle FGH as the polygon ABCDE is to the polygon FGHIK; but the triangle ABC is to the triangle FGH as AB^2 is to FG^2; therefore the similar polygons are as the squares of their homologous sides.

166. Corollary.—If three similar figures have their homologous sides equal to the three sides of a right-angled triangle, the figure made on the side opposite to the right angle shall be equal to the other two.

THEOREM 68.

167. In any triangle, ABC, the square of AB, opposite to one of the acute angles, is equal to the difference between the sum of the squares of the other two sides, and twice the rectangle $BD \times DC$, made by the perpendicular AD, to the side BC.



There are two cases, according as the perpendicular falls within or without the triangle. In the first case, $BD = BC - CD$; and, in the second case, $BD = CD - BC$.

In either case .... $BD^2 = BC^2 + CD^2 - 2BC \times CD$.

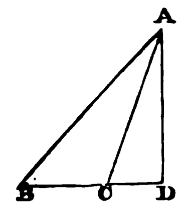
But (160) .... $AB^2 = AD^2 + BD^2$

and (160) .... $AD^2 + CD^2 = AC^2$.

Therefore, by addition, ... $AB^2 = AC^2 + BC - 2B \times CD$.

THEOREM 69.

168. In any obtuse-angled triangle, the square of the side opposite to the obtuse angle is equal to the sum of the squares of the other two sides, and twice the rectangle, $BC \times CD$, made by the perpendicular, AD, upon the side BC.



For .... $BD = BC + CD$;

Therefore, ... $BD^2 = BC^2 + CD^2 + 2BC \times CD$;

But, (160) ...... $AB^2 = AD^2 + BD^2$

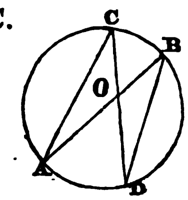
and (160).. $AD^2 + CD^2 = AC^2$

Therefore, by adding these three equations together,

$$AB^2 = AC^2 + BC^2 + 2BC \times CD.$$

THEOREM 70.

169. If any two chords, in a circle, cut each other, the rectangle of the segments of the one is equal to the rectangle of the segments of the other.



Let AB and CD cut each other in O, then $OA \times OB = OD \times OC$.

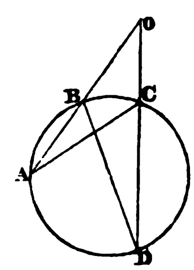
For, join AC and BD: then, in the triangles AOC, BOD, the vertical angles at O are equal; also the angle $A = D$ and $C = B$ (92), consequently the triangles AOC and DOB are similar, and their homologous sides proportional.

Whence $AO : OC :: DO : OB$

Wherefore $AO \times OB = OD \times CO$.

THEOREM 71.

170. If any two chords, in a circle, be produced to meet each other, the rectangle of the two distances, from the point of intersection to each extremity of the one chord, is equal to the rectangle of the two distances from the point of intersection to each extremity of the other chord.



Let AB and CD be two chords, and let them be produced to meet in O; $OA \times OB = OD \times OC$.

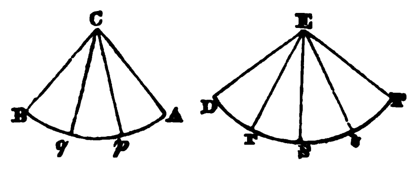
For, join AC and BD; then, in the triangles AOC and DOB the angle at O is common, and the angle $A = D$ (92); therefore the third angle, ACB, of the one triangle, is equal to the third angle, DBO, of the other; consequently the triangles AOC and DOB are similar, and their homologous sides proportional.

Whence $AO : OC :: DO : OB$

Wherefore, $AO \times OB = OD \times OC$.

THEOREM 72.

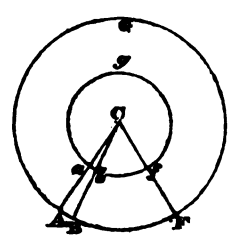
171. In the same circle, or in equal circles, any angles, ACB, DEF, at the centres, are to each other as the arcs AB, DF, of the circles intercepted between the lines which contain the angles.



Let us suppose that the arc AB contains three of such parts as DF contains four. Let A*p*, *pq*, *q*B, be the equal parts in AB, and D*r*, *rs*, &c. the equal parts in DF: draw the lines C*p*, C*q*, E*r*, E*s*, &c.; the angles AC*p*, *p*C*q*, *q*CB, DE*r*, &c. are all equal; therefore, as the arc AB contains $\frac{1}{4}$th part of the arc DF three times, the angle ACB will evidently contain $\frac{1}{4}$ of the angle DEF also three times; and, in general, whatever number of times the arc AB contains some part of the arc DF, the same number of times will the angle ACB contain a like part of the angle DEF.

THEOREM 73.

172. In two different circles, similar arcs are as the radii of the circles.



Let the circles AFG, *afg*, be each described from the centre C. Draw the radii CA, CF, then the arcs A*f* and *af* are similar. Draw CB, indefinitely, near CA, and the sectors C*ab*, CAB, will approach very nearly to isosceles triangles, which are similar to each other; therefore, C*a* : CA :: *ab* : AB; let BF be divided into small arcs, each equal to AB, and draw the radii from each point of division; then *bf* will contain as many arcs, each equal to *ab*, as the arc BF contains arcs equal to AB; therefore *af* is the same multiple of *ab* that AF is of AB; whence C*a* : CA :: *af* : AF.

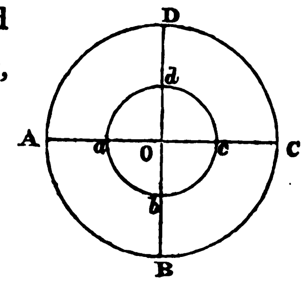
THEOREM 74.

173. The circumference of circles are to one another as their diameters.

For, let the circumferences ABCD, *abcd*, be divided into quadrants by the radii O*a*A, O*b*B, O*c*C, O*d*D, then the quadrants AB, *ab*, will be similar arcs;

therefore, OA : O*a* :: AB : *ab*

wherefore, OA : O*a* :: 4AB : 4*ab*.



PRACTICAL GEOMETRY.

PROBLEM 1.

174. To make an angle at a given point, E, (*fig.* 35, *pl.* I,) in a straight line, DE, equal to a given angle ABC.

From the centre B, with any radius, describe an arc *gh*, cutting BA at *g*, and BC at *h*; from the point E, with the same radius, describe an arc, *ik*, cutting ED at *i*: make *ik* equal to *gh*, and through the point *k* draw EF: then the angle DEF will be equal to the given angle ABC.

PROBLEM 2.

175. To bisect a given angle ABC (*fig.* 36, *pl.* I).

From BA and BC cut off B*e* and B*f*, equal to each other; from the points *e* and *f*, as centres, with any radius greater than the distance *ef*, describe arcs, cutting each other at G, and join BG, which will bisect the angle ABC, as required.

PROBLEM 3.

176. Through a given point *g*, (*fig.* 37. *pl.* I,) to draw a straight line parallel to a given straight line, AB.

From *g* draw *ge*, to cut AB at any angle in the point *e*: in AB take any other point *f*; make the angle B*fh* equal to *feg*, and make *fh* equal to

eg, and through the points *g* and *h* draw the line CD; then CD will pass through *g* parallel to AB, as required.

PROBLEM 4.

177. At a given distance, parallel to a given straight line, AB, (*fig.* 38, *pl.* I,) to draw a straight line, CD.

In the given straight line AB, take any two points, *e* and *f*; and, from the centres *e* and *f*, with the given distance, describe arcs at *p* and *q*; draw the line CD, to touch the arcs *p* and *q*; then CD will be parallel to AB, at the distance required.

PROBLEM 5.

178. To bisect a given straight line, CD, (*fig.* 39, *pl.* I,) by a perpendicular.

From the points C and D, with any distance greater than the half of CD, describe arcs cutting each other in A and B: join AB, and this line will bisect AB perpendicularly.

PROBLEM 6.

179. From a given point C, (*fig.* 40, *pl.* I,) in a given straight line, AB, to erect a perpendicular.

In the straight line, AB, take any two points, *e* and *f*, equally distant from C: from the points *e* and *f*, with any equal radius, greater than the half of *ef*, describe arcs cutting each other at D, and draw CD, which will be perpendicular to AB.

PROBLEM 7.

180. From a given point, B, (*fig.* 1, *pl.* II,) at the extremity of a given straight line, AB, to draw a perpendicular.

Take any point, E, above the line AB, and, with the radius BE, describe the arc *d*BC, cutting AB in *d*: draw the straight line *d*EC, and join BC, which will be the perpendicular required.

PROBLEM 8.

181. From a given point C, (*fig.* 2, *pl.* II,) to let fall a perpendicular to a given straight line, AB.

From the point C, with any radius greater than the distance of AB, describe an arc cutting AB at *e* and *f*; from the points *e* and *f*, as centres, with any equal radius greater than the half of A, describe arcs cutting each other in D, and draw CD, which will be the perpendicular required.

PROBLEM 9.

182. To describe the segment of a circle, which shall have a given length or chord, AB, (*fig.* 3, *pl.* II,) and a given breadth, or *versed sine*, CD.*

By problem 2, bisect the straight line AB, by a perpendicular CE; from the point D, where the perpendicular cuts the chord AB, make DC equal to the breadth, or versed sine: join AC; and, by problem 1, make the angle CAE equal to the angle ACE: from E, as a centre, with the radius EA or EC, describe the arc ACB, which will be the segment required.

PROBLEM 10.

183. Through three given points, A, B, C, (*fig.* 4, *pl.* II,) to describe the circumference of a circle.

Join AB, BC; and, by problem 2, bisect each of the lines AB and BC by a perpendicular, and let the perpendiculars meet each other in I: from the centre I, with the distance IA, IB, or IC, describe the circle ABC, which is that required.

PROBLEM 11.

184. Upon a given straight line, AB, (*fig.* 5, *pl.* II,) to describe an equilateral triangle.

From the centres A and B, with the radius AB, describe arcs cutting each other at C. Join AC and BC; then ABC will be the equilateral triangle required.

PROBLEM 12.

185. Upon a given straight line, AB, (*fig.* 6, *pl.* II,) to describe a square.

From the point B, by problem 5, draw BC perpendicular to AB; make BC equal to AB: from the points A and C, as centres, with a radius equal to AB

* The meaning of *sine*, *versed sine*, &c. is given in Trigonometry, hereafter.

or B, describe arcs cutting each other in D, and join AD and DC; then, ABCD is the square required.

PROBLEM 13.

186. Upon a given straight line, AB, (*figures* 7 and 8, *pl.* II,) to describe a regular polygon of any number of sides.

Produce the side AB to P, and on AP, from the centre B, describe a semi-circle ACP; divide the semi-circumference ACP into as many equal parts as the number of sides intended; through the second division, from P, draw the line BC; bisect AB and BC by perpendiculars cutting each other in S; from S, with the radius AS, BS, or CS, describe a circle ABCDE, then carry the side AB or BC round the remaining part of the arc, which will be found to contain the remaining sides of the number required.

Figure 7 is an example of a pentagon. *Figure* 8 is an example of a hexagon: but, in this figure, we need not proceed by the general method; we have only to make a radius of the given side AB; and take the points A and B as centres; and form the arcs AG and BG, and strike a circle with the radius GA or GB, which will contain the side AB six times.

PROBLEM 14.

187. In a given square, ABCD, (*fig.* 9, *pl.* II,) to inscribe a regular octagon, so that four alternate sides of the octagon may coincide with four sides of the square.

Draw the diagonals AC and BD, cutting each other in S; on the sides of the square make AL, AF, BE, BH; CG, CK; and DI, DM, each equal to half the diagonal; join ME, FG, HI, KL; then will FGHIKLMEF be the octagon required.

PROBLEM 15.

188. In a given triangle ABC, (*fig.* 10, *pl.* II,) to inscribe a circle.

Bisect any two angles, A and B, by the straight lines AE and BE, and the point E, the intersection of these two lines, will be the centre of the inscribed circle: draw ED perpendicular to AB, cutting AB in D; from E, with the

radius ED, describe the circle DFG, which will be inscribed in the triangle ABC, as required.

PROBLEM 16.

189. A circle, DEF, (*fig.* 11, *pl.* II,) and a line AB, touching it, being given, to find the point of contact.

From the centre C draw the perpendicular CD, cutting AB in D, which is the point of contact required.

PROBLEM 17.

190. Two straight lines, AB, BC, (*fig.* 12, *pl.* II,) forming any angle, being given, to describe a circle to touch each of these lines at a given point, A, in one of them.

Make BC equal to BA, and draw AD perpendicular to AB, and CD perpendicular to BC; from the point of intersection D, with the radius DA or DC, describe the circle ACE, which is that required.

PROBLEM 18.

191. In a given circle, ABCD, (*fig.* 13, *pl.* II,) to inscribe a square.

Draw the diameters AC and BD at right angles, and join AB, BC, CD, DA; then ABCD will be the square required.

PROBLEM 19.

192. To describe a segment, ABC, of a circle, by means of an angle.

Let AC (*fig.* 14, *pl.* II,) be the length or chord, and DB the versed sine. Join BA and BC; produce BA to E, and BC to F, making BE and BF of any length, not less than the chord AC. Prepare two straight edges, BE and BF, and fasten them together at the angle B, so that their outer edges may form the angle ABC; and, to keep them to the extent, fix another slip, GH, to each straight edge at G and H. Bring the angular point B to A, then move the angle thus formed by the straight edges, so that the edge BE may always move upon the point A, and the edge BF upon the point C; then if, during the time of moving, a pencil be held to the angular point B, and the point to trace over the plane, the segment of a circle will be described.

ANOTHER METHOD.

193. Let AC (*fig.* 15, *pl.* II,) be the length or chord, and BD the versed sine. Join AB, and draw BE parallel to AC, making BE of any length, not less than AB. Form a triangular piece of wood, ABE: bring the angular point B, of the triangle, to the point A; and move the triangle, so that the side BA may slide upon A, and the side BE upon B: then if, during the motion, a pencil be held at the angular point B, with its point tracing over the plane, the arc AB will be described by the point of the pencil. The arc AB being described, the arc BC will be described in a similar manner; and, consequently, the whole segment of the circle, as required to be done.

PROBLEM 20.

194. Between two straight lines, E and F, (*fig.* 1, *pl.* III,) to find a mean proportional.

Draw the straight line AB. Make AC equal to E, and CB equal to F. Upon AB, as a diameter, describe the semi-circle ADB: from the point C draw CD, perpendicular to AB, and CD will be the mean proportional required.

PROBLEM 21.

195. To find a straight line equal in length, nearly, to the arc of a circle.

Let ABC, (*fig.* 2, *pl.* III,) be the given arc. Join AC, which prolong to F. Bisect the arc ABC in B, and make AE equal to twice AB. Divide CE into three equal parts, and set one of them off from E to F; then the straight line AF is nearly equal in length to the arc ABC.

PROBLEM 22.

196. To describe a triangle, of which the three sides shall be equal to three given straight lines, provided that any two of them are greater than the third.

Let D, E, F, (*fig.* 3, *pl.* III,) be the three given straight lines. Draw AB, and make AB equal to the straight line D. From the point A, with the distance of the line F, describe an arc; and from the point B, with the extent of

the line CE, describe another arc, cutting the former at C, and join AC and BC: then is ABC the triangle required.

In this manner a triangle may be made equal to another given triangle; for this is only making the sides of the triangle equal to those of the given triangle.

PROBLEM 23.

197. To describe a trapezium equal and similar to a given trapezium.

Let it be required to describe a trapezium equal and similar to the given trapezium, ABCD (*fig.* 4, *pl.* III).

Draw the straight line, FG, *fig.* 5, and make FG equal to BC: upon FG describe the triangle FGH, equal to the triangle BCD; and, upon FH, describe the triangle FHE, equal to the triangle BDA, and the whole figure, EFGH, will be equal and similar to the figure ABCD.

PROBLEM 24.

198. To make a rectangle equal to a given triangle.

It is required to make a rectangle equal to the given triangle, ABC (*fig.* 6, *pl.* III).

Draw CF perpendicular to AB, cutting AB in F. Divide CF into two equal parts in the point G. Through G draw DE parallel to AB, and draw AD and BE perpendicular to AB; then the rectangle ABED will be equal to the triangle ABC, as required to be done.

PROBLEM 25.

199. To make a square equal to a given rectangle.

Let it be required to make a square equal to the given rectangle, ABCD, (*fig.* 7, *pl.* III).

Produce the side AB of the rectangle to E, and make BE equal to BC. Draw BG perpendicular to AE; and, on AE, as a diameter, describe the semicircle AGE; and, on the straight line, BG, describe the square BGFH; which is the thing required to be done.

We now see that a triangle may be reduced to a rectangle, and a rectangle may be reduced to a square; therefore a triangle may be reduced to a square.

PROBLEM 26.

200. To make a square equal to two given squares (*fig.* 8, *pl.* III).

Draw the straight line, AB, and BC perpendicular to AB. Make AB equal to the side of one of the given squares, and BC equal to the side of the other, and join AC: then the square described upon AC will be equal to the sum of two other squares described upon AB, BC: Thus, on AB describe the square ABGF; on BC describe the square BHIC; and on AC describe the square ACDE; then will the square ACDE, described upon the hypothenuse, be equal to the two squares ABFG, CBHI, described upon the legs.

PROBLEM 27.

201. To describe a square equal to any given number of squares (*figures* 9, 10, *pl.* III).

Let it be required to make a square equal to the three given squares, A, B, C. Make AB, *fig.* 10, equal to the side of the square A. Prolong BA to F, and draw BG perpendicular to FB, and on BG make BC equal to the side of the square B, and join AC; then a square described on AC is equal to the two squares A and B. Make BG equal to AC, and FB equal to the side of the square C, and join FG, and on FG describe the square FGHI; then the square FGHI is equal to the squares of FB and BG; but the square of FB is equal to the square of C; and, since AC is equal to the squares of A and B, the square of BG will be equal to the squares of A and B; therefore the square FGHI is equal to the three given squares, A, B, C.

PROBLEM 28.

To divide a straight line in the same proportion as another is divided.

202. *Method the first.*—Let AB, (*fig.* 11, *pl.* III,) be a given straight line, divided into the parts AF, FD, DB; it is required to divide another straight line in the same proportion.

Draw AC, making any angle with AB; and make AC equal to the length of the line proposed to be divided. Join BC, and draw the straight lines DE, FG, parallel to BC, cutting AC in G and E; then will the line AC be di-

vided in the same proportion as the straight line AB; as was required to be done.

203. *Method the second.*—Let BC, (*fig.* 1, *pl.* IV,) be the given line. On BC describe the equilateral triangle BCA. Make, on the sides of this triangle, AD, AE, each equal to the length of the line to be divided, and join DE; then DE will be equal to AD or AE: let the given line BC be divided in the points *f, g, h, i.* Draw A*f,* A*g,* A*h,* A*i,* cutting DE in the points *f, g, h, i.* Then the line DE will be divided by these points in the same proportion as the given line BC is divided by the points *f, g, h, i.*

PROBLEM 29.

204. To describe an octagon whose opposite sides are at a given distance from each other (*fig.* 2, *pl.* IV).

Describe a square, ABCD, of which each side is equal to the distance of the opposite sides of the octagon. Draw the diagonals AC and BD, cutting each other in P. From each of the points, ABCD, with a radius equal to AP, BP, CP, or DP, one half of the diagonal, describe the arcs LPF, EPH, GPK, IPM, cutting AB, BC, CD, DA, in the points E, F, G, H, I, K, L, M, and join FG, HI, KL, ME; then will EFGHIKLM be the octagon required.

PROBLEM 30.

To describe an ellipse to any length and breadth.

205. *Method the first* (*fig.* 16, *pl.* II).—Draw the line AC, and make AC equal to the length required; bisect AC by a perpendicular BD, and make EB and ED each equal to half the breadth.

To find any point, *g*, in the curve; with the difference of ED and EA, as a radius, from any point *f*, in EB, describe an arc, cutting EC in *h*. Draw *f h*, and produce it to *g*, and make *hg* equal to EB or ED; then *g* will be a point in the curve, as required.

In the same manner we may find as many points in the curve as we please.

206. *Method the second* (*fig.* 17, *pl.* II).—Divide AE and AF each into the same number of equal parts, as here into five. Through the points of section

1, 2, 3, &c. draw the lines B*h*, B*i*, B*k*, &c.; and through the points of section, 1, 2, 3, &c. in AF, draw the lines 1D, 2D, 3D, &c., cutting the former lines drawn from B, in the points *h, i, k*, &c.; then through the points A, *h, i, k*, &c. draw a curve, and we shall have the fourth part, or quarter, of the whole curve. In the same manner the other quarter DC may be found.

And, by taking the point D, instead of B, and by describing the rectangle upon AC, so that the opposite side may pass through B, and dividing and drawing lines in the same manner, we shall have the whole curve.

207. *Method the third, with a string* (*fig.* 5, *pl.* IV).—Having placed the axes at right angles, and bisecting each other, from one extremity, C, of the conjugate or shorter axis, with AE or BE, half of the transverse, or longer axis, describe arcs, cutting AB at F, *f*; then F, *f*, are the two focal points. Round the points F, *f*, stretch a string, and fasten the ends of it; carry the point forward, as to H, keeping the string always tense, and the point H, &c. will describe the curve as required.

PROBLEM 31.

208. To represent an ellipse by means of the arcs of circles (*fig.* 6, *pl.* IV).

Let AB be the length, and CD the breadth, as before. Draw BE perpendicular to CD. Make BF equal to EC. Bisect BF in *f*, and *f*C and ED, cutting each other in *g*. Bisect *g*C, by a perpendicular, cutting CD produced in *i*. Make E*k* equal to E*i*, and join F*i*, cutting AB in *l*. Make E*m* equal to E*l*. Through *m* draw *kp*, and *iq*; and, through *l*, draw *kr*, *ih*. From *i*, with the radius *i*C, describe an arc, *qh*; and, from *k*, with the same radius, describe an arc *pr*. From *m*, with the radius *mp*, describe an arc *p*A*q*; and, from *l*, with the same radius, describe an arc *r*B*h*: then will the compound figure, A*q*, C*h*, B*r*, D*p*, represent the ellipse required.

This method is used only for describing an arc on paper: it may also be used for drawing the lines, perpendicular to a real elliptic arch, for the joints of the stones. The first method of describing the curve may be used occasionally, from want of a proper instrument; but, of all the methods, none is so accurate, in practice, as the trammel, which is next described.

PROBLEM 32.

209. To describe an ellipse with a trammel (*fig.* 4, *pl.* IV).

Draw the straight line AB, and make AB equal to the length of the given ellipse. Bisect AB in E, by the perpendicular CD. Make ED and EC each equal to half the lesser axis. Let the trammel-rod be *fgh*. Suppose the nut *f* to be fixed, and the nuts *g* and *h* to be moveable. Move the nut *g* so that the distance *fg* may be equal to EC or ED; and move the nut *h*, so that the distance *fh* may be equal to EA or EB: then set *g* and *h* in the grooves of the trammel, and, by moving the end of the rod *f*, so that the pin *g*, may slide along the line AB, and the pin *h* along the line CD; the point or pencil will trace out the curve of the ellipse as required.

PROBLEM 33.

210. A rectangle being given, to describe an ellipse, so that the two axes may have the same proportion as the sides of the rectangle (*fig.* 3, *pl.* IV).

Let ABCD be the given rectangle; it is required to describe an ellipse that shall pass through the points A, B, C, D, and of which the axes shall have a ratio equal to the sides AB, BC, or to CD, DA, of the rectangle.

Draw the diagonals AC, BD, cutting each other in I; and, through the point I, draw EG, parallel to AB or DC, cutting one of the sides BC in P; and HF, parallel to AD or BC, cutting one of the sides AB in K. From I, with the radius IK, describe an arc, KL, cutting IG in L. Bisect the arc KL in M, and draw *m*N, parallel to EG, cutting the diagonal DB in N. Join NP and NK, and draw BF, cutting HF in F, and draw BG, cutting EG in G. Make IE equal to IG, IH equal to IF; then EG and FH are the two axes of the ellipse, which will be described as in the following problem.

PROBLEM 34.

211. To describe an *hyperbola*, or a figure that may have any curvature at the summit, that we please (*fig.* 18, *pl.* II).

Let AC be the base, or what is called a *double ordinate:* make ED equal to the height, in the middle of AC; then, upon AC, as a side, describe a rectangle,

AFGC, so that the opposite side may pass through D; produce ED to the point B; take the point B, farther or nearer from D, according as the curvature at D is required to be flatter or quicker; and observe that, the quicker the curve is at D the flatter it will be towards A and C. The point B being thus fixed, divide AE into any number of parts, as here into four; also, divide AF into the same number of parts, viz. four; through 1, 2, 3, &c., the points of section in AE, draw lines to B; and through the points of section, 1, 2, 3, &c. in AF, draw lines to D, cutting the former lines drawn to B, in the points *h*, *i*, *k*, &c.; and, through the points A, *h*, *i*, *k*, &c., draw a curve, which will be the half of an hyperbola, or an hyperbolic curve.

PROBLEM 35.

212. To describe a *parabola* upon a given ordinate, AE, and a given abscissa, ED* (*fig.* 19, *pl.* II).

Make EC equal to EA, and complete the rectangle AFGC; so that the opposite side may pass through D. Proceed as in the two former problems, 33 and 34; excepting that, instead of drawing the lines to B, through the points 1, 2, 3, &c., in AE, to draw them parallel to ED.

PROBLEM 36.

213. To describe the figure of the Sines † (*fig.* 20, *pl.* II).

Describe the quadrant FHG, equal to the height of the figure, and divide the arc HG into any number of equal parts; the more of these the more perfect the operation will be; and extend the chords to double the number of parts upon the line AC, which is a continuation of FH, and mark the points of division. Draw the lines 1*k*, 2*l*, 3*m*, &c. perpendicular to AC; and, from the points 1, 2, 3, &c. of division in the quadrant, draw lines 1*k*, 2*l*, 3*m*, &c. parallel to AC, and through the points A, *k*, *l*, *m*, &c., draw a curve, which will be the figure of the sines, as required.

* The PARABOLA is a figure arising from the section of a cone, when cut by a plane parallel to one of its sides. This, with other terms in Conics, is fully described, under CONIC SECTIONS, hereafter.

† The SINE, or *right sine*, of an arch, is a right line drawn from one end of that arch, perpendicular to a radius, drawn to the other end of the same.—See TRIGONOMETRY, hereafter.

GEOMETRY OF SOLIDS.

DEFINITIONS OF SOLIDS.

214. A Right Cylinder is that which is formed by the revolution of a rectangle about one of its sides; the line round which the rectangle revolves is called the *axis* (plural *axes);* and the circles generated by the two opposite sides of the rectangle, perpendicular to the axes, are termed the ends or bases. The surface of the cylinder, generated by the line parallel to the axis, is termed the *curved surface,* which is either straight or convex, according as a straight edge is applied, parallel to the axis, or in any other direction.

215. A Right Cone is that which is formed by supposing a right-angled triangle to revolve about one of its legs or perpendicular sides; the fixed leg, or line, is called the *axis;* the surface generated by the other leg is called the *base;* and the surface formed by the hypothenuse, or side opposite the right angle, is denominated the *curved surface,* which is either straight or convex, according as a straight edge is applied upon the surface from the vertex, or in any other direction.

216. A Sphere or Globe is that which is formed by supposing a semi-circle to revolve upon its diameter; the diameter upon which the semi-circle revolves is called the *axis,* and the surface formed by the arc of the semi-circle is called the *curved surface,* which is convex, in whatever way it may be tried by a straight edge.

217. An Ellipsoid is formed or generated by supposing a semi-ellipse to revolve upon one its axes; the axis thus fixed is called the *axis of the ellipsoid,* and the surface generated by the curve is termed the *curved surface.*

PROBLEM 37.

218. To describe a conic section, from the cone, through a line given in position, in the section passing through the axis.

Let ABC, (*figures* 1, 2, 3, *pl.* VI,) be the section of a right cone, and let DE be the line of section. Through the apex or top of the cone, C, draw CF, parallel to the base AB of the section, and produce ED to meet AB in D, as in *figures* 2 and 3, or AB produced in G, as in *fig.* 1, as also to meet CF in F. On AB describe a semi-circle, which will be equal to half the base of the cone. In the semi-circle take any number of points, *a, b, c,* &c. Draw D*d,* in *figures* 2 and 3, and G*d* in *fig.* 1, perpendicular to AB and G*d'*, in *fig.* 1, perpendicular to GF; as, also, D*d'*, *figures* 2 and 3, perpendicular to DF. From the points *a, b, c,* &c. draw lines, *ae, bf, cg,* &c., cutting G*d* (*figure* 1) and D*d* (*figures* 2 and 3) in the points *e, f, g,* &c. In *figure* 1, make in G*e'*, G*f'*, G*g'*, &c. equal to G*e*, G*f*, G*g*, &c.; and in D*d'*, (*figures* 2 and 3,) make D*e'*, D*f'*, D*g'*, &c. equal to D*e*, D*f*, D*g*, &c. Through the points *e'*, *f'*, *g'*, &c. draw lines to F. Through the points *a, b, c,* &c. draw lines perpendicular to AB. From the points of section, in AB, draw lines to the vertex C of the cone, cutting the sectional line, DE, in *l, m, n,* &c. Through the points of section, *l, m, n,* &c., draw *lh, mi, nk,* &c. perpendicular to DE. Through the points D, *h, i, k,* &c. in *fig.* 1, or *d'*, *h, i, k,* &c. *figures* 2 and 3, draw a curve, which will be the conic section required.

OBSERVATIONS.

219. In the first of these figures, the line of section cuts both sides of the section of the cone; in this case, the curve D*hik* and *e*E is an *Ellipse.* In *fig.* 2, the line of section DE is parallel to the side AC of the section of the cone; in this case, the curve *d' hi*, &c. E, is a *Parabola.* In *fig.* 3, the line of section, DE, is not parallel to any side of the cone; it must, therefore, when produced with the sides of the section through the axis, meet each of these two sides in different points: in this case, the section *d'*, *h, i,* &c., E, is either an *Ellipse* or

Hyperbola; but the case is determined to be an hyperbola by the line of section meeting the opposite side BC at AC, where it cuts above the vertex at the point B′.

Here we may observe, that the line of section, DE, is the same as that which has before been called the *abscissa,* the part EB produced, contained between the two sides of the section, is called the *axis major ;* and the line D*d*, perpendicular to DE, an *ordinate*.

Hence the same section may be found by the method already shown in the problem; viz. by drawing any straight line, *deb*, *fig.* 4: make *de* equal to DE, *fig.* 3, and *eb* equal to EB, *fig.* 3. Through *d* draw the straight line DD at right angles to *db′*: make *d*D equal to D*d′*, *fig.* 3; then, with the axis major, b′*e*, the abscissa *ed*, and the ordinate *d*D, on each side of the abscissa describe the curve of the hyperbola, which will be of the same species as that shown in *fig.* 3.

PROBLEM 38.

220. To describe a cylindric section, through a line given in position, upon the section passing through its axis (*fig.* 4, *pl.* VI).

This is no more than a particular case of the last problem. For a cylinder may be considered as a cone, having its apex at an infinite distance from its base; or, practically, at a vast distance from its base. In this case all the lines, for a short distance, would differ insensibly from parallel lines; and this is the construction shown at *fig.* 5, which is therefore evident. But as the section of a cylinder so frequently occurs, I shall here give a more practical description of it. Thus—

Let ABHI be a section of a right cylinder, passing through its axis, AB being the side which passes through the base, and let DE be the line of section. On AB describe a semi-circle; and, in the arc, take any number of points, *a*, *b*, *c*, &c. from which draw lines perpendicular to the diameter, AB, cutting it in Q, R, S, &c: perpendicular to AB, or parallel to AI or BH, draw the lines Q*q*, R*r*, S*s*, &c. cutting the line of section, DE, in the points, *q*, *r*, *s*, &c.: from the points of section, *q*, *r*, *s*, &c. draw the lines *qi*, *rk*, *sl*, &c. per-

pendicular to the line of section, DE. Make the ordinates *qi*, *rk*, *sl*, &c. each respectively equal to the ordinates Q*a*, R*b*, S*c*, &c.; and through the points D, *i*, *k*, *l*, &c. to E, draw a curve, which will evidently be the section of the cylinder, as required.

The same may be done in this manner, viz.—Bisect the line of section DE in the point *t*. Draw *tm* perpendicular to DE. Make *tm* equal to the radius of the circle which forms the end of the cylinder; then, with the *axis major*, DE, and the semi-axis minor, *tm*, describe a semi-ellipse, which will be the section of the cylinder required.

A DEFINITION.

221. A Cuneoid is a solid ending in a straight line, in which, if any point be taken, a perpendicular from that point may be made to coincide with the surface: the end of the cuneoid may be of any form whatever.

The cuneoid, which occurs in architecture, has a semi-circular or a semi-elliptical end, parallel to the straight line to which the perpendicular is applied.

PROBLEM 39.

222. To find the section of a cuneoid, with a semi-circular base, the given *data* being a section through the axis, perpendicular to the vertex, or sharp end, and the line of section upon that end.

Let ABC, (*fig.* 6, *pl.* VI,) be the section through the axis, perpendicular to the sharp edge, and let DE be the line of section.

This construction is similar to that of finding the section of a cylinder, excepting that, instead of drawing parallel lines from the base, AB, they are, in this figure, drawn from the points of section in AB to the point C, which is the vertex of the cuneoid: the ordinates, Q*a*, R*b*, S*c*, &c., being transferred, respectively, to *qi*, *rk*, *sl*, &c.; and the curve D, *i*, *k*, *l*, &c. to E, being drawn through the points, D, *i*, *k*, *l*, &c., by hand.

PROBLEM 40.

223. Given the position of the seats of three points, in the circumference of the base of a cylinder, and the lengths of the perpendiculars intercepted

between the points and their seats, to find the section of the cylinder passing through these three points.

Through the three points, A, B, C, (*fig.* 7, *pl.* VI,) describe the circumference of a circle. Join the two remote points, A and B, and draw AD, CF, and BE, perpendicular to AB. Make AD equal to the height upon A, BE equal to the height upon B, and CF equal to the height upon C. Produce BA and ED to meet each other in H: draw CG parallel to BH, and FG parallel to EH. Join GH. In GH take any point, G, and draw GK perpendicular to CG, cutting BH in K: from the point K draw KI, perpendicular to EH, cutting EH in L. From H, with the radius HG, describe an arc, cutting KI at I. Join HI. In the circumference, ACB, take any number of points *a, b, c,* &c., at pleasure, and draw *ae, bf, cg,* &c., parallel to GH, cutting AB at *e, f, g,* &c. Through the points *e, f, g,* &c., draw lines *ei, fk, gl,* &c., parallel to GK, or AD, or BE, cutting DE at *i, k, l,* &c.; from the points of section, *i, k, l,* &c., draw the lines *in, ko, lp,* &c., parallel to HI. Transfer the ordinates, *ea, fb, gc,* &c., to *in, ko, lp,* &c.; then, through the points D, *n, o, p,* &c. draw the curve D*nop*, &c. to E, and it will be the section cut by the plane, as required.

PROBLEM 41.

224. Given the great circle of a sphere, and the line of position of a section at right angles to that great circle, to find the form of the section.

Let ABC, (*fig.* 8, *pl.* VI,) be the great circle, and AB the line of section.

On AB, as a diameter, describe a semi-circle, which will be the section required: since all the sections of a sphere, or globe, are circles.

PROBLEM 42.

225. Given the section of an ellipsoid, passing through the fixed axis, and the line of position of another section, at right angles to the first section, to find the form of the section through that line.

Let ABCD, (*fig.* 9, *pl.* VI,) be the section through the fixed axis, and EF the line of position. Through the centre of the ellipsoid draw AC parallel to EF. Bisect EF in H, and draw HG perpendicular to EF. Find HG a fourth

proportional to AC, DB, HE. Then, with the axis major, EF, and the semi-axis minor, HG, describe a semi-ellipse, and it will be the section of the ellipsoid required.

If AC be the axis major, BD will be the axis minor. In this case, join DC, and draw EG parallel to DC; then HG will be the height found geometrically.

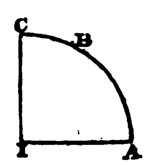
PROBLEM 43.

226. To find the section of a cylindric ring, perpendicular to the plane passing through the axis of the ring, the line of section being given.

Let ABED, (*fig.* 10, *pl.* VI,) be the section of the ring, passing through its axis, and let AB be a straight line, passing or tending to the centre of the two concentric circles, AD and BE; also, let DE be the line of section. On AB describe a semi-circle, and take *a*, *b*, *c*, &c., any number of points in its circumference; draw the ordinates, *ae*, *bf*, *cg*, &c. Through the points, *e*, *f*, *g*, &c., in the diameter AB, draw the concentric circles, *ei*, *fk*, *gl*, &c., cutting the sectional line DE in the points *i*, *k*, *l*, &c. Through the points, *i*, *k*, *l*, &c., draw *in*, *ko*, *lp*, &c., perpendicular to DE; transfer the ordinates *ea*, *fb*, *gc*, &c., of the semi-circle, to *in*, *ko*, *lp*, &c.: and, through the points D, *n*, *o*, *p*, &c. draw the curve, D*nopq*E, which is the section required.

PLANE TRIGONOMETRY.

DEFINITIONS OF TERMS IN TRIGONOMETRY.*

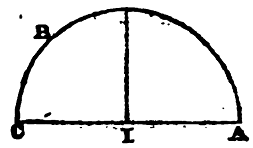


227. THE COMPLEMENT OF AN ARC is the difference between that arc and a quadrant or quarter of a circle.

Thus, the arc BC, which is the difference between AC and AB, is the complement of AB; and AB is, in like manner, the complement of BC.

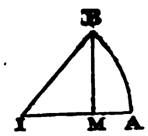
* Trigonometry is that branch of Geometry which treats exclusively on the properties, relations, and measurement, of triangles.

228. The Supplement of an Arc is the remainder between that arc and a semi-circle.

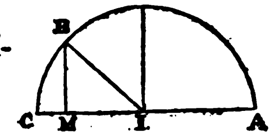


Thus, the arc given being AB, its supplement is BC.

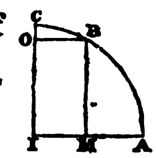
229. The Sine of an Arc is a straight line, drawn from one extremity of the arc, upon and perpendicular to a radius or diameter.



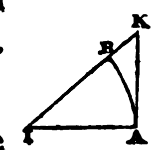
Thus, BM is the *sine* of the arc AB; and here it is evident that an arc and its supplement have the same sine.



230. The Co-sine of an Arc is the sine of the complement of that arc. Hence, BO or IM is the co-sine of the arc AB; and, therefore, the sine of the complement BC.

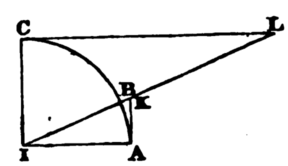


231. The Tangent of an Arc is a straight line, drawn from one extremity of the arc, where it touches it, to meet the prolongation of the radius through the other extremity.

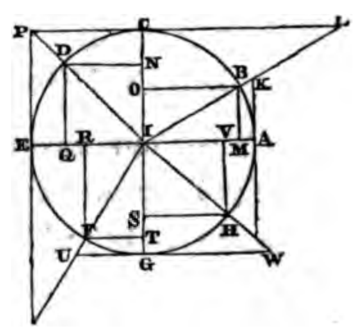


The line AK, touching the arc at A, and extended to meet the radius IB produced, is the tangent of the arc AB.

232. The Co-tangent of an Arc is the tangent of the complement of that arc.



Thus, CL is the co-tangent of the arc AB, or the tangent of the arc BC.



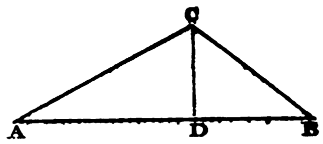
In the annexed diagram let AB, AC, AD, AE, AF, AG, AH, to A, be the several portions of the circumference, by supposing the point B to revolve round the circumference from A to B, C, D, E, F, G, H, the sine of any arc, in the first quadrant, increases from A to C, where it is the greatest possible, and then decreases to E, where it becomes zero; the sine will, therefore, be positive for the first semi-circumference, and in the other half it will be negative.

The *co-sine* will be positive in the first quarter, negative in the second and third, and again positive in the fourth.

The *tangent* will be positive in the first quarter, negative in the second, positive in the third, and negative in the fourth.

TRIGONOMETRY.—THEOREM 1.

233. If a perpendicular be drawn from an angle of a triangle, to the opposite side, which is the base; then, as the base is to the sum of the two sides, so is the difference of the sides to the difference of the segments of the base.



For, (*theorem* 62, *page* 56) .... $AC^2-CD^2=AD^2$

and, again, (*theorem* 62) .... $BC^2-CD^2=BD^2$.

Subtract the second equation from the first, and the result is .... $AC^2-BC^2=AD^2-BD^2$:

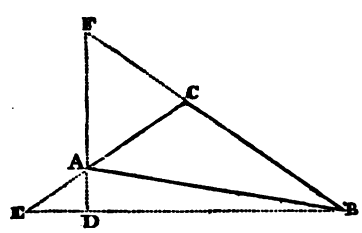
but, since the difference of the squares of any two quantities is equal to a rectangle contained by their sum and difference;

therefore.................... $(AC+BC)(AC-BC)=(AD+BD)(AD-BD)$

Whence, (*theorem* 40, *page* 41) $AD+BD : AC+BC :: AC-BC : AD-BD$.

TRIGONOMETRY.—THEOREM 2.

234. The sum of the two sides of a triangle is to their difference as the tangent of half the sum of the angles at the base is to the tangent of half their difference.



Let ABC be a triangle; then, of the two sides, CA and CB, let CB be the greater. Produce CA to E, and make CE = CB, and join BE. Produce BC to F; and, through A, draw FD, perpendicular to EB, meeting it in D; then FBD will be half the sum of the angles at the base, and ABD half their difference. Likewise, DF is the tangent of the angle FBD, and AD the tangent of the angle ABD: moreover BF is the sum of the two sides BC, CA, and AE is their difference.

Then, by similar triangles, BFD, EAD,.... $BF \times AD = FD \times EA$. Wherefore $BF : AE :: FD : AD$; which is the proposition to be demonstrated.

CONIC SECTIONS.

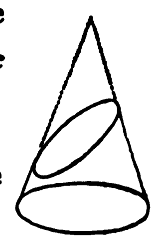
DEFINITIONS OF CONIC SECTIONS.

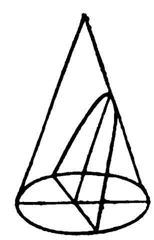
235. A Cone is a solid body, terminating in a point, called its vertex, and having a circle for its base, connected to the vertex by a curved surface, which every where coincides with a straight line passing through its vertex, and through any point in the circumference of the base. If a cone be cut by an imaginary plane, the figure of the section so formed acquires its name according to the inclination or direction of the cutting plane.

236. A plane passing through the vertex of a cone, and meeting the plane of the base, is called a *directing plane*, and the line of common section is called a *directing line*.

237. If a cone be cut by a plane parallel to the directing plane, the section is denominated a *conic section*.

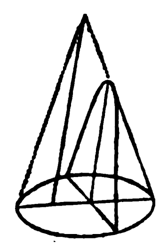
238. If the directing line fall without the base of the cone, the section is called an *ellipse*.





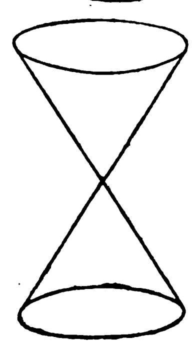
239. If the directing line touch the circumference of the base, the section is called a *parabola*.

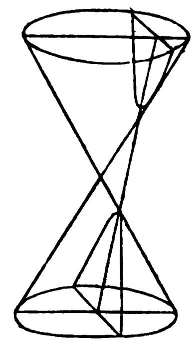
240. If the directing line fall within the base, the section is called an *hyperbola*.



241. Equal opposite cones are those which have their axes in the same straight line; and, if cut by a plane through their common line of axis, the sides of the section will be two straight lines cutting each other.

Hence the two equal and opposite cones join each other at their vertices, and have their vertical angles equal.





242. If the plane which produces the section of an hyperbola be extended so as to cut the opposite cone, the two sections are denominated *opposite hyperbolas.*

243. If the plane of a conic section be cut perpendicularly by another plane, which passes through the axis of the cone, the line of common section, in the plane of the figure, is called the *primary line.*

244. A point where the primary line cuts a conic section is called a *vertex* of that conic section.

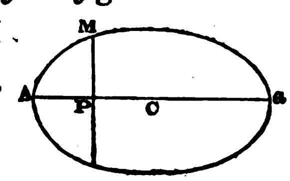
Hence the ellipse has two vertices, opposite hyperbolas have each one, and the parabola has one.

OF THE ELLIPSE.

245. That portion of the primary line terminated at each extremity by the vertices of the curve, is called the *axis major,* or *transverse axis.*

246. A straight line, drawn perpendicularly to the axis major, from any point in it, to meet the curve, is called an *ordinate.*

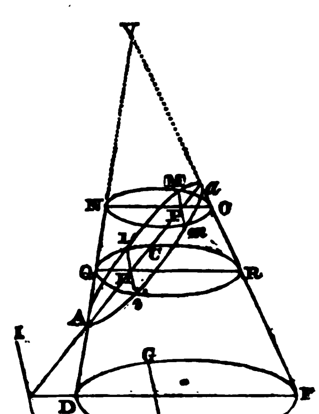
247 The middle of the axis major is called the *centre of the figure.*



In the figure here annexed, Aa is the axis major, PM an ordinate to it, and the point C, in the middle of Aa, is the centre of the ellipse, AMa.

THE ELLIPSE.—THEOREM 1.

248. The squares of the ordinates of the axis are to each other as the rectangles of the segments of the axis, from each ordinate to each of the two vertices of the curve.



Let VDF be a plane passing through the axis of the cone, perpendicular to the cutting plane of the section A1M*ami*, and let A*a* be their common section, meeting the conic surface in the points A, *a*; (then A*a* will be the axis major,) and let Q1R*i*, NMO*m*, be sections of the cone, parallel to the base. Then, because the base DGFE is perpendicular

to the plane VDF, the three sections, A1M*ami*, Q1R*i*, NMO*m*, are all perpendicular to the plane VDF; therefore their common sections, 1*i*, M*m*, are perpendicular to the plane VDF, and to the lines A*a*, QR, NO; but, because the plane VDF passes through the axis of the cone, it will divide all the circles parallel to the base into two equal parts; therefore QR and NO are the diameters of the circles Q1R*i*, NMO*m*; and, because the chords 1*i*, M*m*, are perpendiculars to the diameters, QR, NO, they will be bisected; let H be the point of bisection in 1*i*, and P the point of bisection in M*m*.

Let $CA = Ca = a$, $CP = x$, $PM = y$, $CH = z$, $HI = \gamma$, $PN = t$, $PO = u$, $HQ = v$, and $HR = w$.

$$\text{Then } AP = CA + CP = a + x,\ aP = Ca - CP = a - x,$$
$$\text{and } AH = CA - CH = a - z,\ aH = Ca + CH = a + z.$$

$$\text{Now, by similar triangles, } \begin{cases} APN,\ AHQ \ldots v(a+x) = t(a-z) \\ aPO,\ aHR \ldots w(a-x) = u(a+z) \end{cases}$$

$$\text{and, by the circle, } \ldots\ldots \begin{cases} \ldots\ QIR\ \ldots\ldots\ \gamma^2 = vw \\ \ldots\ NMO\ \ldots\ldots\ tu = y^2. \end{cases}$$

Wherefore, eliminating t, u, v, w, by multiplying the given equations, the result will be $\gamma^2(a+x)(a-x) = y^2(a-z)(a+z)$, or, by actual multiplication, $\gamma^2(a^2 - x^2) = y^2(a^2 - z^2)$.

249. Corollary 1.—Hence, every chord perpendicular to the axis major is bisected by the axis major.

250. Corollary 2.—Hence the tangents at the extremity of the axis major are perpendicular to the axis major.

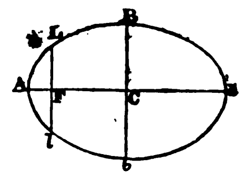
DEFINITIONS RELATIVE TO THE ELLIPSE, CONTINUED.

251. A straight line drawn through the centre, perpendicularly to the axis major, and terminated by the curve, is called the *axis minor*, or *conjugate axis*.

252. A third proportional to the axis major and minor, is called the *parameter*, or the *latus rectum* of the axes.

Thus a and b being the semi-transverse and semi-conjugate axes, $2a : 2b :: 2b : p$, the parameter; therefore, $ap = 2b^2$, or if $f = \frac{1}{2}p$, we shall have $af = b^2$, therefore $f = \frac{b^2}{a}$

253. That point in the axis, cut by an ordinate, which is equal to half the parameter, is called the *focus*.



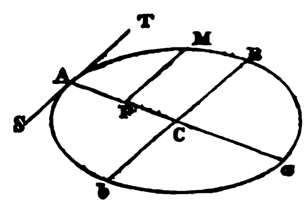
In the figure here annexed, B*b*, drawn through C, is the semi-axis minor; and, if L*l* be a third proportional to Aa, B*b*, then L*l* is the parameter, and the point F, where it cuts Aa, is the focus.

254. Any line drawn through the centre, and terminated at each extremity by the curve, is called a *diameter*.

255. A diameter, which is parallel to a tangent at one extremity of another diameter, is called a *conjugate diameter* to that other diameter.

256. A straight line, parallel to a tangent, at the extremity of any diameter, terminated at one extremity by that diameter and the curve at the other, is called an *ordinate* to that diameter.

257. The portion of a diameter between the centre and an ordinate, is called the *abscissa* of that ordinate, or of that diameter.

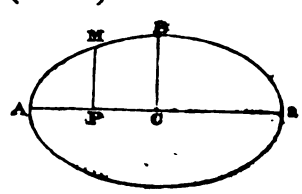


In the figure here annexed, the straight line, A*a*, drawn through the centre, C, is a diameter; and, if ST be a tangent at A, and the diameter B*b* be drawn parallel to ST, the diameter is called the *conjugate diameter* of A*a*; and PM, parallel to ST or B*b*, is an *ordinate* to the diameter A*a*; and the distance CP, on the diameter A*a*, is called the *abscissa*.

ELLIPSE.—THEOREM 2.

258. The square of the axis major is to that of the axis minor as the rectangle contained by the two parts of the axis major, from the ordinate to each vertex, to the square of the ordinate.

From the preceding theorem, $\gamma^2(a^2-x^2)=y^2(a^2-\varkappa^2)$.



Now let b represent the semi-axis minor, and let the ordinate γ become b, then will its abscissa, $\varkappa$, become zero, and, consequently, $\gamma^2(a^2-x^2)=y^2(a^2-\varkappa^2)$ will become $b^2(a^2-x^2)=a^2y^2$; whence $a^2 : b^2 :: a^2-x^2 : y^2$.

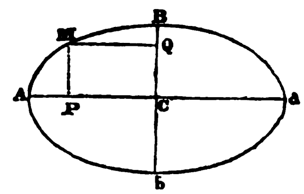
259. **Corollary 1.**—Hence every ellipse has two focii at an equal distance from the centre; because $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$.

260. **Corollary 2.**—Hence the tangent at either vertex of the curve is parallel to the ordinates; and, consequently, perpendicular to the axis major.

ELLIPSE.—THEOREM 3.

261. The square of the axis minor is to that of the axis major as the rectangle contained by the two parts of the axis minor, from the ordinate to the extremity of the axis minor, to the square of the ordinate.

For, by *theorem* 2, ... $a^2y^2 = b^2(a^2 - x^2)$
and, by transposition,. ... $b^2x^2 = a^2(b^2 - y^2)$
therefore, $b^2 : a^2 :: b^2 - y^2 : x^2$.

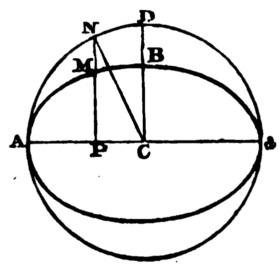


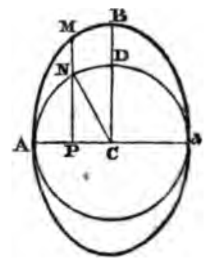
262. **Corollary 1.**—Hence the tangent at each extremity of the axis minor is parallel to the ordinates; and, consequently, parallel to the axis major.

263. **Corollary 2.**—Hence the axis major and minor are reciprocally conjugate diameters.

264. **Corollary 3.**—If a circle be described on either axis of an ellipse, an ordinate of the circle will be to the corresponding ordinate of the ellipse as the axis of this ordinate is to the other axis.

Let PM cut the inscribed circle in N, and let PM be produced to cut the circumscribing circle in N; in both cases let CP $= x$, PM $= y$, PN $= \gamma$, CA $= a$, and CB $= b$.





By *theorem* 2, (258,) .... $b^2(a^2 - x^2) = a^2y^2$,
and, by the right-angled triangle CPN, ... $\gamma^2 = a^2 - x^2$.

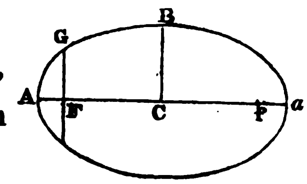
Therefore, multiplying these two equations, we have $b^2\gamma^2 = a^2y^2$, and extracting the root $b\gamma = ay$; wherefore $a : b :: \gamma : y$.

265. Corollary 4.—Hence any two corresponding ordinates of the circle and ellipse are in the same constant ratio of the two axes.

ELLIPSE.—THEOREM 4.

266. The square of the distance of the focus from the centre is equal to the difference of the squares of the semi-axes.

$$CF^2 = AC^2 - BC^2$$



Let the ordinate FG, which passes through the focus, be denoted by f, and CF, the distance of the focus from the centre, by ε.

Then, by the equation of co-ordinates.... $a^2y^2 = b^2(a^2 - x^2)$

and, since (252) AC : BC :: BC : FG, .. $a^2f^2 = b^4$.

Now, in the first of these two equations, when the abscissa x becomes ε, the ordinate y will become f, and, consequently, $a^2f^2 = b^2(a^2 - \varepsilon^2)$. Whence $b^4 = b^2(a^2 - \varepsilon^2)$ or $b^2 = a^2 - \varepsilon^2$; and, by transposition, $\varepsilon^2 = a^2 - b^2$.

267. Corollary 1.—Hence, because $af = b^2$, therefore $af = a^2 - \varepsilon^2$.

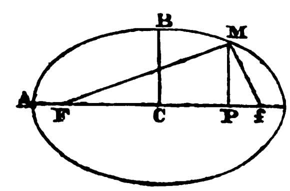
268. Corollary 2.—Hence $b^2 = a^2 - \varepsilon^2 = a^2 - c^2a^2 = a^2(1 - c^2)$.

269. Corollary 3.—Because $a^2y^2 = b^2(a^2 - x^2)$ and that $b^2 = a^2(1 - c^2)$. Therefore, by substitution, there will arise $y^2 = (1 - c^2)(a^2 - x^2) = a^2 - x^2 - c^2a^2 + c^2x^2$.

270. Corollary 4.—The semi-conjugate axis CB is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices, for $b^2 = a^2 - \varepsilon^2 = (a + \varepsilon)(a - \varepsilon)$.

ELLIPSE.—THEOREM 5.

271. The sum of two lines drawn from the focii, to meet any point in the curve, is equal to the transverse axis.



Let $FM = R$, $fM = r$, $FC = fC = \varepsilon = ca$.

Then will $FP = CF + CP = ca + x$,

and..... $fP = Cf - PC = ca - x$.

By Geom. (160).... $FM^2 = PM^2 + FP^2$, $R^2 = y^2 + c^2a^2 + 2cax + x^2$,

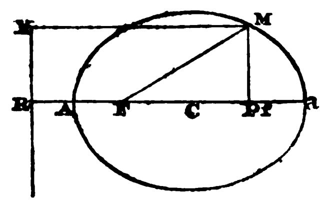
and, by *theorem* 4, *cor.* 3, ... $y^2 = a^2 - c^2a^2 + c^2x^2 - x^2$.

Wherefore, eliminating y, by adding these equations together, we have the equations .... $R^2 = a^2 + 2cax + c^2x^2$.

Then, extracting the roots of each side of this equation, we have $R = a + cx$. In the same manner will be found $r = a - cx$; therefore $R + r = 2a$.

272. Corollary 1.—Hence $R = a + \frac{\epsilon}{a}x$, and $r = a - \frac{\epsilon}{a}x$, for $\epsilon = ca$ or $c = \frac{\epsilon}{a}$.

273. Corollary 2.—Because $R = a + \frac{\epsilon}{a}x$, we shall have $aR = a^2 + \epsilon x$; and, by transposition, $a(R - a) = \epsilon x$, hence $R - a : x :: \epsilon : a$.



274. Corollary 3.—Since the ratio of ϵ to a, or that CF to CA is constant, if we produce aA to R, and find the point R by making CF : CA :: CA : CR, and draw RX perpendicular to aR, and MX parallel to aR, and let CR $= d$; then will $R - a : x :: a : d$; wherefore $Rd - ad = ax$, or, by transposition, $Rd = ad + ax = a(d + x)$. Therefore $d + x : R :: d : a$; wherefore MX is to MF always in the constant ratio of CR to CA; and hence we have another method of constructing the ellipse.

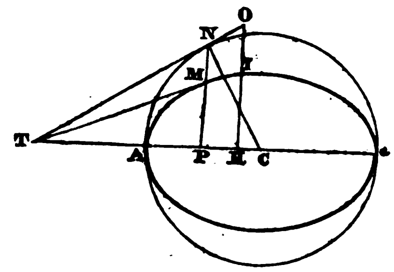
N.B. The line RX is called the directrix. Several writers make this the fundamental principle from which all the other properties emanate.

275. Corollary 4.—Hence CR : CA :: aR : aF; because, when the point M comes to a, the line MX or PR will become aR, and the radius vector, FM, will become Fa.

276. Corollary 5.—Hence CA : CF :: aR : aF.

ELLIPSE.—THEOREM 6.

277. The tangents from the corresponding points in the curves of a circle and ellipse, made by the prolongation of an ordinate of the ellipse, will meet the axis major produced in the same point T.



Let ANa be a circle described upon the axis major, and let PN be a tangent to the circle in N. Join TM; then, if TM does not touch the ellipse, let it cut it in M, I; and, through I, draw the ordinate HO, meeting the ellipse in I, the circle in L, and the tangent in O.

By similar triangles .... $\begin{cases} \text{TPM, THI} \;\ldots\ldots\ldots\; TP \times HI = PM \times TH \\ \text{TPN, THO} \;\ldots\ldots\ldots\; PN \times TH = TP \times HO \end{cases}$

and, by *theorem* 3, *cor*. 4, .. $PN : PM :: HL : HI \therefore PM \times HL = PN \times HI$.

Therefore, by multiplication, we shall find $HL = HO$, which is impossible; therefore TM does not cut the ellipse, and it must, in consequence, be a tangent at M.

ELLIPSE.—THEOREM 7.

278. In the straight line, Ta, of the axis major, the semi-axis, CA, is a mean proportional between the abscissa, CP, and the distance, CT, from the centre to the intersection of the tangent.

For the triangle CNT (see the preceding diagram) is right-angled at N, since PN meets the circle at N; and, because PN is perpendicular to TC, we shall have,

by the similar triangles PCN, NCT, ... $CP \times CT = CN^2$

and by the circle .... $CN^2 = CA^2$.

Therefore, by multiplication, $CP \times CT = CA^2$.

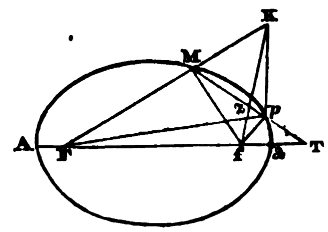
If $CT = u$, this conclusion, analytically expressed, is $ux = a^2$; or, if s be the subtangent, then will $u = s + x$; and, therefore, $(s + x)x = a^2$, that is, $sx + x^2 = a^2$.

279. COROLLARY 1.—Whence, if z be any other abscissa, and S its subtangent, and if $s + z = v$, then $vz = a^2$; and, since $ux = a^2$, therefore $ux = vz$.

280.—COROLLARY 2.—When the abscissa becomes equal to the excentricity, x becomes ϵ; and, consequently, $ux = a^2$ becomes $u\epsilon = a^2$, or $S\epsilon + \epsilon^2 = a^2$.

ELLIPSE.—THEOREM 8.

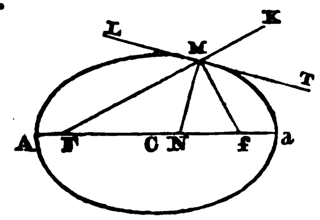
281. The line bisecting the angle at any point in the curve, formed by one of the lines drawn from one of the focii, and the prolongation of the line drawn from the other focus, will be a tangent to the curve.



Let MT bisect the angle fMK; then, if MT is not a tangent, it will cut the ellipse; let p be the point where it meets the curve, and, making MK$=$Mf, join fK, fp, Fp, Kp, and let MT cut Kf, in x.

Then, in the triangles fMx, KMx, fM being $=$KM, Mx common, and the included angles fMx, KMx, equal; the base fx is $=$Kx, and the angle fxM$=$KxM.

Again, in the triangles fpx, Kpx, fx being $=$Kx, px common, and the angles fxp, Kxp, equal, the base fp is $=$Kp. Therefore Fp+pK$=$Fp+pf= FM+Mf=FM+MK=FK; wherefore the sum of the two sides, Fp, pK, of the triangle FpK, is equal to the third FK; which is impossible; therefore as MT cannot cut the ellipse: it must be a tangent.

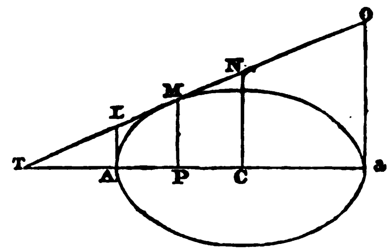


282. Corollary.—Hence, because the angle KMf is bisected, the angle FMf will also be bisected by the normal MN. For since the angle TMf is $=$TMK, and TMK equal to the opposite angle FML; therefore TMf is $=$ FML: and, because the angles TMN and LMN are each a right angle; therefore, taking away the angles TMf, LMF, which are equal, from the right angles TMN, LMN, there will remain the angle NMf equal to the angle NMF.

ELLIPSE.—THEOREM 9.

283. If there be any tangent meeting the four perpendiculars to the transverse axis, the one at the extremity of the transverse, next to the point of intersection, will be to that which terminates in the point of contact, as that which passes through the centre is to the remaining one at the other extremity of the transverse.

AL : PM :: CN : aO

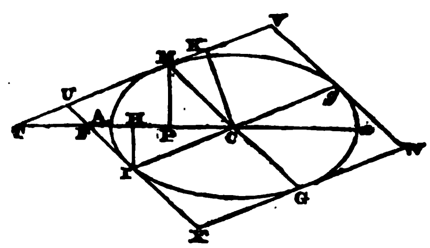


Let AL$=k$, PM$=l$, CN$=m$, and aO$=n$. Then, if we can show that the distances TA, TP, TC, T, are four proportionals, AL, PM, CN, aO, being the homologous sides of similar triangles, will likewise be proportionals.

By *theorem* 7, (278,) $sx+x^2=a^2$; therefore, by transposition, $sx+x^2-a^2=o$; to each side of this equation add s^2+sx, and we have $s^2+2sx+x^2-a^2=s^2+sx$; that is, because $s^2+2sx+x^2$ is a complete square, and that $s^2+sx=s(s+x)$ $(s+x)^2-a^2=s(s+x)$, that is, $(s+x+a)(s+x-a)=s(s+x)$. Whence $s+x-a : s :: s+x : s+x+a$.

ELLIPSE.—THEOREM 10.

284. Every parallelogram, UVWX, circumscribing an ellipse, having its sides parallel to two conjugate diameters, is equal to the rectangle of the two axes.



If tangents be drawn at the extremities of two conjugate diameters, they form an ellipse, of which CMUI is a fourth part.

Draw CK, perpendicular to MT. Let C$t=v$, TM$=t$, CI$=n$, CT$=u$, CK$=p$, PT$=s$.

By *theorem* 7, (278) .... $a^2-x^2=sx$

and by *theorem* 7, *cor*. 1, (279) .... $ux=v\varkappa$

and by similar triangles $\begin{cases} t\text{CI, CTM} \ldots\ldots & vt=nu \\ \text{ICH, MTP} \ldots\ldots & ns=t\varkappa \end{cases}$

and by *theorem* 2, (258) .... $a^2\gamma^2=b^2(a^2-\varkappa^2)$

and by similar triangles, TCK, CIH .... $p^2n^2=u^2\gamma^2$

and by *theorem* 7, (278,)................ $u^2x^2=a^4$.

For, multiplying the first four equations, $a^2-x^2=\varkappa^2$; or, by transposition, $a^2-\varkappa^2=x^2$. Multiply this and the three remaining equations and $pn=ab$.

285. Corollary.—Hence $a^2=x^2+\varkappa^2$, and $b^2=y^2+\gamma^2$.

ELLIPSE.—THEOREM 11.

286. The sum of the squares of any pair of conjugate diameters is equal to the sum of the squares of the two axes.

Let CM$=m$, CI$=n$. See the preceding figure.

By *theorem* 10, *cor.* (285) .... $\begin{cases} \ldots a^2 = x^2 + z^2 \\ \ldots b^2 = y^2 + \gamma^2 \end{cases}$

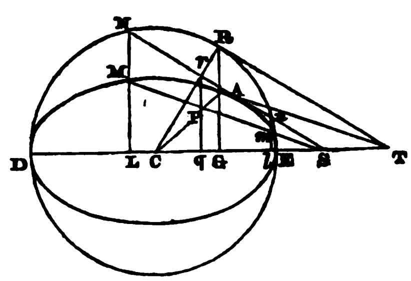
and by *Geometry*, (160) .... $\begin{cases} x^2 + y^2 = m^2 \\ z^2 + \gamma^2 = n^2. \end{cases}$

Therefore, by adding these together, we have $a^2 + b^2 = m^2 + n^2$.

N.B. All these theorems concerning the Ellipse, and their demonstrations, are in the very same words as the corresponding number of those for the Hyperbola, next following; having sometimes only the word sum *changed for the word* difference.

ELLIPSE.—THEOREM 12.

287. If a chord be bisected, the tangent at the extremity of the diameter, passing through the point of bisection, will be parallel to that chord.



Through M, m, draw LN, ln, to meet the circumference of the circle described on the axis DE, in the points, N, n. Produce Mm and DE, to meet each other in S, and join Sn, SN, and let the chord Mm be bisected in P.

Then, because Sl : lm :: SL : LM

and .... lm : ln :: LM : LN

therefore eliminating LM, lm, ... Sl : ln :: SL : LN.

Therefore Sn and SN are in a straight line. Through P draw qr, meeting DE in q, and Nn in r; then Nn is bisected in r. Join Cr, and produce it to meet the circumference in R, and draw the tangent RT, meeting DE produced in T; then RT is parallel to NS. Draw RG, perpendicular to DE, meeting DE in G, and CA in A.

Then, because LN : LM :: qr : qP

qr : qP :: GR : GA

therefore .... LN : LM :: GR : GA

And, therefore, the point A is in the curve of the ellipse. Then drawing AT, AT is a tangent by the preceding proposition.

Now, because CR : Cr :: CT : CS

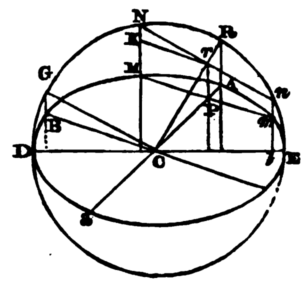
and .... CR : Cr :: CA : CP

therefore ... CT : CS :: CA : CP.

Therefore AT is parallel to Mm.

ELLIPSE.—THEOREM 13.

288. The rectangle of the squares of any semi-diameter, and of an ordinate to it, is equal to the rectangle of the square of the semi-conjugate and the difference of the squares of the semi-diameter of the abscissa, and of the abscissa itself.



$$CA^2 \times PM^2 = BC^2 \times (CA^2 - CP^2).$$

Draw rK parallel to PM, cutting MN in K, and draw CG parallel to Nn, cutting the circumference in G, and CB parallel to PM, cutting the ellipse in B, and join BG.

Let CR$=r$, C$r=u$, rN$=rn=v$, CA$=a$, CB$=b$, CP$=x$, PM$=r$K$=y$.

By the equation of the circle.................. $v^2=r^2-u^2$

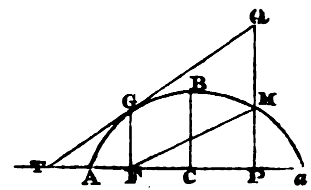
By similar triangles, $\begin{cases} \text{BCG, K}r\text{N} \ldots\ldots\ldots & r^2y^2=b^2v^2 \\ \text{ACR, PC}r \ldots\ldots\ldots & rx^2=a^2u^2. \end{cases}$

Multiply the first and second equations together, and $r^2y^2=b^2r^2-b^2u^2$; or, by transposition, $b^2u^2=r^2(b^2-y^2)$. Multiply this and the third equation, and $b^2x^2=a^2(b^2-y^2)$; or, by transposition, $a^2y^2=b^2(a^2-x^2)$.

ELLIPSE.—THEOREM 14.

289. The semi-ordinate, together with its prolongation to meet a tangent at the extremity of a *latus rectum*, is equal to the *radius vector* through the same focus with that of the latus rectum.

FM=PQ.



Let CT$=u$, PQ$=v$, PT$=w$, and FT$=q$.

Then will FT $=$ CT $-$ CF $= u - \varepsilon = q$

and..... PT $=$ CT $+$ CP $= u + x = w$.

By notation .... $\begin{cases} u - \varepsilon = q \\ w = u + x \end{cases}$

and by *theorem* 7, *cor.* 2,.............. $u\varepsilon = a^2$

and by *theorem* 4, *cor.* 1,.............. $af = a^2 - \varepsilon^2$

and by similar triangles, TPQ, TFG,.... $qv = fw$.

Multiply the first and second of these equations by ε, and the results are $u\varepsilon - \varepsilon^2 = \varepsilon q$, and $w\varepsilon = u\varepsilon + \varepsilon x$. In each of these two equations, for $u\varepsilon$ substitute its equal a^2 from the second, and we have $a^2 - \varepsilon^2 = \varepsilon q$ and $w\varepsilon = a^2 + \varepsilon x$. Multiply these two equations, and the remaining two of the given equations, and the final result is $av = a^2 + \varepsilon x$, therefore $v = a + \frac{\varepsilon}{a}x$.

But, by *theorem* 5, we have $R = a + \frac{\varepsilon}{a}x$, therefore $R = v$.

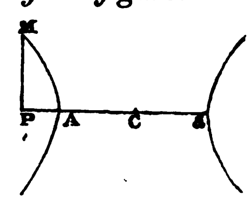
OF THE HYPERBOLA.

DEFINITIONS RELATIVE TO THE HYPERBOLA.

290. That portion of the primary line between the vertices of the two opposite curves, is called the *transverse axis*.

291. A straight line, drawn perpendicularly to the transverse axis, between the transverse axis and the curve, is called an *ordinate*.

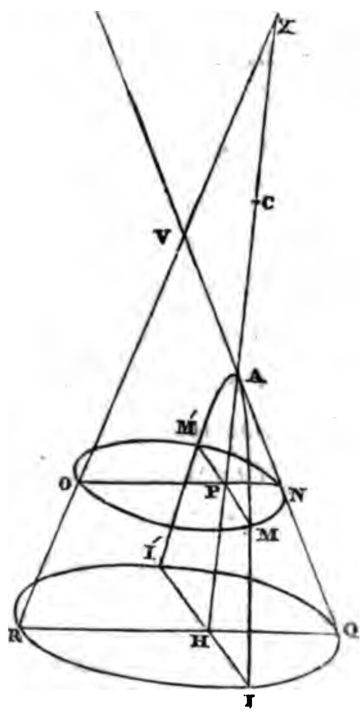
292. The middle of the transverse axis is called the *centre of the figure*.



293. In the annexed diagram, Aa is the transverse axis, C, the middle of Aa, the centre; PM the ordinate, and CP the abscissa.

HYPERBOLA.—THEOREM 1.

294. The squares of the ordinates of the axis are to each other as the rectangles of the two segments of the axis, from each ordinate to each of the two vertices of the opposite curves.



Let VRQ, passing through the common line of axis of two opposite cones, be a plane perpendicular to the cutting plane of the opposite sections; and let AMII'M' be one of the sections, HA the common line of section of the two planes, and let HA cut the two conic surfaces in A, X; then AX will be the primary axis; and let OMNM' be a section of the cone parallel to its base, RIQI'.

Because the plane of the base, RIQI', is perpendicular to the plane VRQ, the plane OMNM' will also be perpendicular to the plane VRQ; and, since each of the two planes AMII'M', RIQI', and AMII'M', OMNM', are perpendicular to the plane VRQ, their common sections II', MM', are perpendicular to the plane VRQ; therefore II', MM', are perpendicular to the lines RQ, ON, AH, in the plane VRQ; and, because the plane VRQ passes through the axis of the cone, it will divide all the circles parallel to the base into two equal parts; therefore RQ, ON, will be diameters of the two circles; and, since the chords II', MM', are at right angles to the diameters RQ, ON, the chords II', MM', will be bisected in H and P. Therefore HI=HI', and PM=PM'.

Let $CA=CX=a$, $CP=CX=x$, $PM=y$, $CH=z$, $HI=\gamma$, $PN=t$, $PO=u$, $HQ=v$, and $HR=w$.

$$\begin{aligned} \text{Then AP} &= \text{CP}-\text{CA}=x-a, \\ \text{XP} &= \text{CP}+\text{CX}=x+a, \\ \text{AH} &= \text{CH}-\text{CA}=z-a, \\ \text{and XH} &= \text{CH}+\text{CX}=z+a. \end{aligned}$$

$$\text{Now, by similar triangles, } \begin{cases} \text{APN, AHQ} \ldots (x-a)v=t(z-a) \\ \text{XPO, XHR} \ldots (a+x)w=u(z+a) \end{cases}$$

$$\text{and, by the circle, } \ldots\ldots \begin{cases} \ldots\ \text{QIRI}' \ldots\ldots\ \gamma^2=vw \\ \ldots\ \text{NMOM}' \ldots\ tu=y^2. \end{cases}$$

Therefore, eliminating t, u, v, w, by multiplying the given equations, there will result $\gamma^2(x-a)(x+a)=y^2(z-a)(z+a)$, or, by actual multiplication, $\gamma^2(x^2-a^2)=y^2(z^2-a^2)$; therefore $\gamma^2 : y^2 :: (z-a)(z+a) : (x-a)(x+a)$.

295. Corollary 1.—Hence $\frac{\gamma^2}{z^2-a^2}=\frac{y^2}{x^2-a^2}$ is a constant quantity.

296. Corollary 2.—Hence if z be made constant, γ will be constant also. Therefore, when z^2-a^2 becomes a^2, let $\gamma=b$, and, consequently, $\frac{b^2}{a^2}=\frac{y^2}{x^2-a^2}$.

297. Corollary 3.—Hence every chord, perpendicular to the transverse axis, is bisected by the same axis.

298. Corollary 4.—Hence the tangent at the extremity of the transverse axis is bisected by the transverse axis.

299. Corollary 5.—Hence $ay^2=b^2(x^2-a^2)=b^2x^2-a^2b^2$.

HYPERBOLA.—DEFINITIONS, CONTINUED.

300. The constant value b of γ, when z^2-a^2 becomes a^2, is called the *semi-conjugate axis*, or twice b the *conjugate axis*.

301. A third proportional to the transverse and conjugate axes, is called the *parameter*, or *latus rectum*.

Thus a and b being the semi-transverse and semi-conjugate axes, $2a:2b::2b:p$, the parameter; therefore, $ap=2b^2$, or if $f=\frac{1}{2}p$, we shall have $af=b^2$, therefore $f=\frac{b^2}{a}$

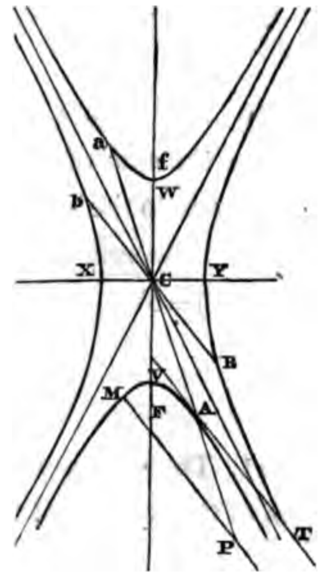
302. That point in the axis cut by an ordinate which is equal to half the parameter is called the *focus*.

303. If there be two opposite hyperbolas, and two others having the same centre, and their common line of axis at right angles to that of the two former, the transverse equal to the conjugate of the two former, and the conjugate equal to the transverse of the two former; these four hyperbolas are called *conjugate hyperbolas*.

304. Any straight line, drawn through the centre, and terminated by opposite curves, is called a *diameter*.

A diameter which is parallel to a tangent, at the extremity of another diameter, is called a *conjugate diameter* to that other diameter.

305. A straight line, parallel to a tangent, meeting the diameter in one extremity, and the curve in the other, is called an *ordinate* to that diameter.

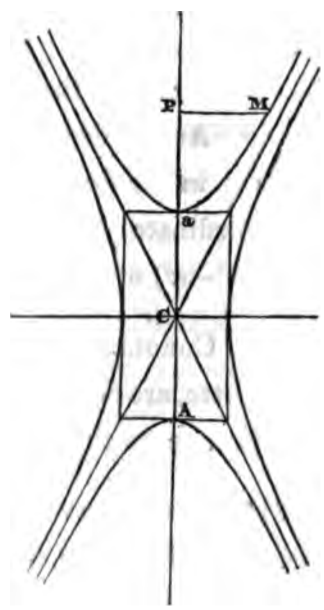


306. The distance between the centre and an ordinate is called the *abscissa*.

In the diagram here annexed, the straight line Aa, drawn through the centre, C, is a diameter; and, if AT is a tangent at A, then Bb, parallel to AT, passing through the centre C, is the conjugate diameter; PM parallel to AT, is the ordinate to Aa, and CP is the abscissa.

HYPERBOLA.—THEOREM 2.

307. The square of the transverse axis is to the square of the conjugate axis as the rectangle of the two distances from the ordinate to the vertex of each curve.



$$CA^2 : CB^2 :: PA \times Pa : PM.$$

For $\frac{b^2}{a^2}=\frac{y^2}{x^2-a^2}=\frac{y^2}{(x-a)(x+a)}$

Therefore $a^2 : b^2 :: (x-a)(x+a) : y^2$.

308. Corollary 1.—Hence every pair of opposite hyperbolas has two focii at an equal distance from the centre; because $y^2=\frac{b^2}{a^2}(a^2-x^2)$: and, since the origin of the abscissa commences at the centre, therefore the ordinate y must be the same at the same distance on each side of the centre.

309. Corollary 2.—Hence the tangent at the vertex of either curve is parallel to the ordinates, and, consequently, perpendicular to the transverse axis.

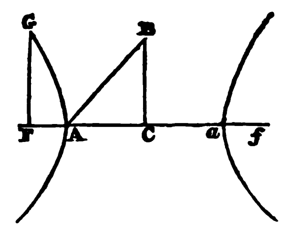
HYPERBOLA.—THEOREM 3.

310. The square of the conjugate axis is to the square of the transverse axis, as the sum of the squares of the semi-conjugate axis; and that of the ordinate is to the square of the abscissa as $CB^2 : CA^2 :: CB^2+PM^2 : CP^2$.

For, (*theorem* 2,) $a^2y^2 = b^2x^2-a^2b^2$; and, therefore, by transposition, $a^2(y^2+b^2)=b^2x^2$, consequently, $b^2 : a^2 :: y^2+b^2 : x^2$. (See *figure, theorem* 2.)

HYPERBOLA.—THEOREM 4.

311. The square of the distance of the focus from the centre is equal to the sum of the squares of the two axes.



$$CF^2=AC^2+BC^2$$

Let the ordinate FG, which passes through the focus, be denoted by f, and CF, the distance of the focus from the centre, by ϵ.

Then, by the equation of the co-ordinates, .. $a^2y^2=b^2(x^2-a^2)$.

And, since (301) .... $a^2f^2=b^4$.

Now, in the first of these equations, when the abscissa x becomes equal to ϵ, the ordinate y will become f; and, consequently, $a^2f^2=b^2(\epsilon^2-a^2)$; whence, $b^4=b^2(\epsilon^2-a^2)$ or $b^2=\epsilon^2-a^2$, and, by transposition, $\epsilon^2=a^2+b^2$.

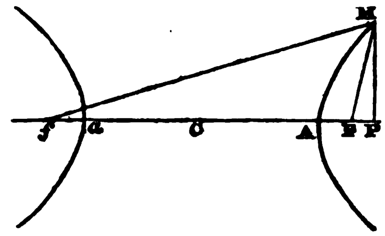
312. Corollary 1.—The two semi-axes, and the distance of a focus from the centre, are the sides of a right-angled triangle, ACB, and the hypothenuse, AB, is equal to the distance of the focus from the centre.

313. Corollary 2.—The conjugate axis, CB, is a mean proportional between AF and Fa, or fa, fA, the distances between either focus and the two vertices; for $b^2=\epsilon^2-a^2=(\epsilon-a)(\epsilon+a)$.

314. Corollary 3.—Hence, if $\epsilon=ca$, then will $y^2=a^2-x^2+c^2x^2-c^2a^2$; for, (*theorem* 2,) $a^2y^2=b^2(x^2-a^2)$. But, by this theorem, we have $b^2=\epsilon^2-a^2$; then, multiplying these two equations, we have $a^2y^2=(\epsilon^2-a^2)(x^2-a^2)$; or, substituting ca for ϵ, we shall have $y^2=(c^2-1)(x^2-a^2)=a^2-x^2+c^2x^2-c^2a^2$.

HYPERBOLA.—THEOREM 5.

315. The difference of two lines, drawn from the focii to meet any point in the curve, is equal to the transverse axis.



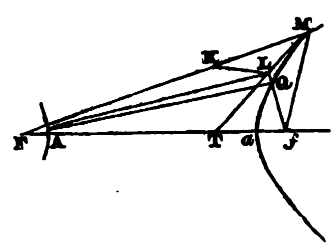
Let $FM = R$, $fM = r$, $FC = fC = \varepsilon = ca$.
Then will $FP = CP + CF = x + ca$,
and ..... $fP = CP - Cf = x - ca$.

By Geometry, (*prop*. 62,) .. $FM^2 = PM^2 + FP^2$, $R^2 = y^2 + x^2 + 2acx + c^2a^2$.

And, by *theorem* 4, *cor*. 3, (314) .... $y^2 = a^2 - x^2 + c^2x^2 - c^2a^2$,
wherefore, eliminating y, by adding these equations together, we have the equation .... $R^2 = c^2x^2 + 2acx + a^2$.
then, extracting the roots of each side of this equation, we have $R = cx + a = \frac{\varepsilon}{a}x + a$. In like manner will be found $r = cx - a = \frac{\varepsilon}{a}x - a$; therefore $R - r = 2a$.

HYPERBOLA.—THEOREM 6.

316. The line bisecting the angle, at any point in the curve formed by the two lines drawn from that point to each focus, is a tangent.



The tangent MT at M will bisect the angle FMf.

In MF take $MK = Mf$, and in MT take any point L; join fL, and let it meet the curve in Q; join also KL, FL, FQ. Then, by hypothesis, the angle $KML = fML$, $KM = fM$, and LM is common, the base LK is $= Lf$; but the difference of any two sides of a triangle is less than the third;* therefore $FL - LK$, or $FL - Lf$, is less than FK, or Ff, or $FM - fM$, or $FQ - fQ$. Hence fL is greater than fQ; for, since $FL - fL$ is less than $FQ - fQ$, if fL were less than fQ, $FL + LQ$

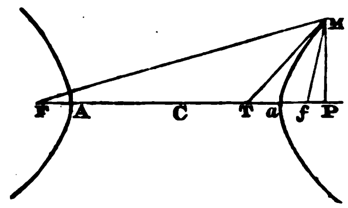
* It is shown by every writer of Elementary Geometry, that the sum of every two sides of a triangle are greater than the third. Let a, b, c, be the three sides of a triangle; then, $a + b \ulcorner c$, $a + c \ulcorner b$, $b + c \ulcorner a$: therefore, by transposition, $a \ulcorner c - b$, $a \ulcorner b - c$, $b \ulcorner c - a$, $b \ulcorner a - c$, $c \ulcorner b - a$, $c \ulcorner a - b$.

N.B. $\ulcorner$ signifies *greater than*.

would be less than FQ; which is impossible. Therefore every point, L, in MT, except M, is without the curve of the hyperbola; and MT touches it at M.

HYPERBOLA.—THEOREM 7.

317. In the line of the axis major, the rectangle contained by the distance between the centre and the intersection of the tangent, and the distance between the centre and the ordinate, is equal to the square of the semi-axis major.



Let $CT = u$.

Then will $FT = CF + CT = \epsilon + u$.

and .... $fT = Cf - CT = \epsilon - u$.

By *theorem* 5, (315,) .... $\begin{cases} R = \frac{\epsilon}{a}x + a \\ \frac{\epsilon}{a}x - a = r \end{cases}$

and by Geometry, (*theorem* 57,) .... $r(\epsilon + u) = R(\epsilon - u)$.

Multiplying these three equations together, we have $ux = a^2$.

HYPERBOLA.—THEOREM 8.

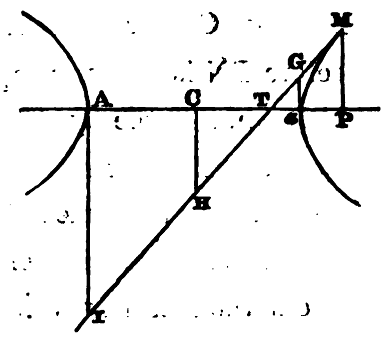
318. The semi-transverse axis is a mean proportional between the two distances in the line of the transverse axis; the one from the centre to the ordinate, and the other from the centre to the intersection of the tangent.

For, by the preceding proposition, $ux = a^2$, therefore $x : a :: a : u$. (See *figure, theorem* 7.)

Or, if the subtangent $PT = s$, then will $CT = CP - PT = x - s$. Whence $(x - s)x = a^2$ or $x^2 - sx = a^2$; expressed as in the same proposition of the ellipse.

HYPERBOLA.—THEOREM 9.

319. If there be any tangent, and four perpendiculars to the line of axis, contained between the tangent and the line of axis, the rectangle under two, which passes through the vertices, will be equal to the rectangle of the third, which passes through the centre, and the fourth which is the ordinate.



$aG \times AI = CH \times PM$

or $aG : PM :: CH : AI$.

PT being $= s$. See the preceding theorem.

Then ...... $aP = CP - CA = x - a$

$u = CT = CP - PT = x - s$

$AT = CA + CP - PT = a + x - s$

$aT = PT - aP = s - x + a$.

Now, by *theorem* 8, $a^2 = x(x-s) = x^2 - sx$. Therefore, by transposition, $a^2 + sx - x^2 = 0$; add $sx - x^2$ to each side of this equation, and $a^2 - x^2 + 2sx - s^2 = sx - s^2$, or $a^2 - (x^2 - 2sx + s^2) = s(x-s)$; and, since the difference of two squares is equal to the rectangle of the sum and difference of their roots, $(a+x-s)(a-x+s) = s(x-s)$.

Whence .... $AT \times aT = PT \times CT$

And, by similar triangles, $\begin{cases} TCH, TAI \ldots CT \times AI = CH \times AT \\ TaG, TPM \ldots aG \times TP = aT \times PM. \end{cases}$

Therefore, eliminating AT, aT, PT, CT, by multiplying these equations and $aG \times AI = CH \times PM$.

HYPERBOLA.—PROBLEM.

320. Given the transverse axis of an hyperbola and an ordinate, to find the conjugate axis and assymtotes, (which are two straight lines, such as, if produced indefinitely with the curve, will never meet each other,) and thence to describe the curve itself.

Let Aa, (*fig.* 1, *pl.* V,) be the transverse axis, and let PM be an ordinate. Make PD equal to AP. Then on aD describe the semi-circle aND. Produce PM to N. Draw AR perpendicular to CD, and make AR equal to CA. Join NR, and produce NR and DA, if necessary, to meet each other in S; and draw MS, cutting AR in Q. Produce QA to T, and make AT equal to AQ. Then QT will be the conjugate axis, or AQ, AT, will each be the semi-conjugate axis. Through the points C, T, draw JH; and through the points C, Q, draw IK: then JH and IK are the *assymtotes*, by which the curve may be described.

Because $CP = x$, and $CA = a$, $AP = PD = x - a$, and aP equal to $x + a$; therefore PN is a mean proportional between $x + a$ and $x - a$, or between aP and PD; but, in the hyperbola, the transverse axis is to the conjugate as the mean proportion between $x + a$, and $x - a$ to the ordinate y or PM; but AR is divided in Q, in the same ratio as PN is in N. Therefore PN : PM :: AR : AQ. That is, as the mean proportional between $x + a$, and $x - a$ is to the ordinate PM, so is the semi-transverse axis to the semi-conjugate axis.

OF THE PARABOLA.

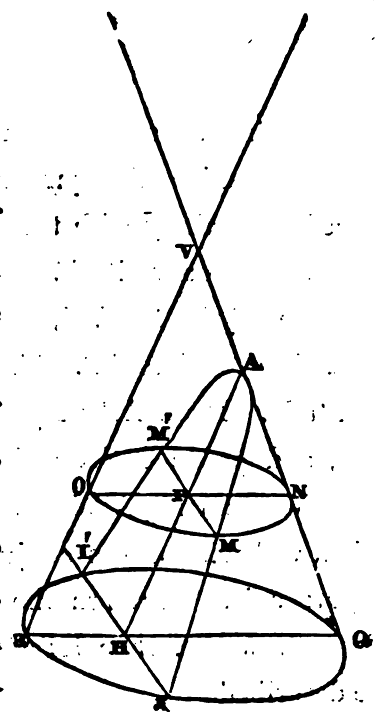
DEFINITIONS RELATIVE TO THE PARABOLA.

321. That portion of the primary line which is within the curve, and which is terminated at one extremity by the vertex, is called the *axis*.

322. A straight line, drawn perpendicularly to the axis, between it and the curve, is called an *ordinate*.

PARABOLA.—THEOREM 1.

323. The squares of the ordinates of the axis are to each other as their distances from the vertex.



Let VRQ be a plane, passing through the axis of the cone, perpendicular to the cutting plane of the section AMII′M′; and let AH be their common section; then AH will be the axis. Let QIRI′ be a section of the cone parallel to the base:

Then, because the base RIQI′ is perpendicular to the plane VRQ, the two sections AMII′M′, OMNM′, are perpendicular to the plane VRQ; therefore their common sections, MM′, II′, are also perpendicular to VRQ and to the lines AH, RQ, ON; but, because the plane VRQ passes through the axis of the cone, it will divide every circle pa-

rallel to the base into two equal parts; therefore ON is a diameter of the circle OMNM′; and, because the chords MM′, II′, are perpendicular to the diameters RQ, ON, they will be bisected; let H be the point of bisection in II′, and P the point of bisection in MM′.

324. Let AP$=x$, PM$=y$, AH$=z$, HI$=\gamma$, HQ$=v$, HR$=$OP$=w$, and PN$=t$.

By similar triangles, .. APN, AHQ......$vx=tz$

and, by the circle, ... QIR...... $\gamma^2=vw$

and, by the circle, ...NMO...... $tw=y^2$.

By multiplying these equations, the result will be $x\gamma^2=zy^2$. Therefore $x : z :: y^2 : \gamma^2$.

325. Corollary 1.—Hence $\frac{y^2}{x}=\frac{\gamma^2}{z}$.

326. Corollary 2.—Hence, because $\frac{y^2}{x}=\frac{\gamma^2}{z}$, whatever may be the values of x, z, y, γ, therefore $\frac{y^2}{x}$ or $\frac{\gamma^2}{z}$ is a constant quantity: hence, putting $\frac{y^2}{x}=\frac{1}{2}p$; therefore $y^2=\frac{1}{2}px$.

PARABOLA.—DEFINITIONS, CONTINUED.

327. The constant quantity p is called the *parameter* or *latus rectum*.

328. Corollary.—Hence the parameter is a third proportional to the distance of the ordinate from the vertex, and the ordinate itself; for $2x : 2y :: 2y : p$, or $px=2y^2$, that is, $y^2=\frac{1}{2}px$.

THE THREE CURVES OF THE CONIC SECTIONS.—PROBLEM.

329. The vertical section of a right cone being given, and the position of the axis of a conic section, to describe that section.

Let AVB, (*fig*. 2, *pl*. V,) be the section of a cone through its axis; let *ig* be the line of the axis, and let it cut the section AVB at *h*, and the opposite side BV, produced, at *g*. On *gh* describe the semi-circle *hqsg*. Draw V*p* parallel to AB, cutting the axis in *p*. Bisect *hg* in *r*, and draw *pq*, *rs*, perpendicular to *hg*. Make *pw* equal to *p*V; then, with the transverse axis, *hg*, and the ordinate, *pw*, describe the ellipse *hwtg*, cutting *rs* at *t*; then *rt* is the semi-conjugate axis.

In *fig*. 3, *pl*. V, draw the line aA, for the transverse axis, equal to *gh*, *fig*. 5; and bisect Aa in C, the centre. Through A draw DE, perpendicular to Aa;

make AD and AE each equal to *rt*, *fig*. 2. Through C and D draw JH, and through C and E draw IK; then JH and IK are the assymtotes.

Draw any line, *ai*, cutting the assymtote IK at *h*, and the assymtote JH at *g*. Make *hi* equal to *ag*, and *i* will be a point in the curve. In the same manner we may find as many more points as we please.

Let the axis be *cf*, *fig*. 2, cutting the sides of the section AV, BV, at *c* and *f*. Draw *cd*, *ef*, parallel to AB, cutting AV at *e* and BV at *d*.

In *fig*. 4, draw AB equal to *cf*, *fig*. 2. Bisect AB in C. Make CF, C*f*, each equal to the half of *df*, or the half of *ce*; then, with the transverse axis AB, and focii F, *f*, describe the ellipse ADBE.

Again, in *fig*. 2, let the axis be *mn*, and let *ms* be parallel to the side AV of the vertical section, cutting the base AB at *m*, and the side BV at *n*. On AB describe the semi-circle AOB, and draw *m*O perpendicular to AB.

In the straight line AA′, *fig*. 5, take any point, D, and make DA, DA′ each equal to *m*O, *fig*. 2. Draw DC, perpendicular to AA′, and make DC equal to *mn*, *fig*. 2. Then, with the abscissa AB, and ordinates DA, DA′, describe the curve ACA′, which will be the parabola.

CHAPTER II.

CARPENTRY.

Carpentry is the art of applying timber in the construction of buildings.

The cutting of the timbers, and adapting them to their various situations, so that one of the sides of every timber may be arranged according to some given surface, as indicated in the designs of the architect, requires profound skill in geometrical construction.

For this purpose it is necessary, not only to be expert in the common problems, generally given in a course of practical geometry, but to have a thorough knowledge of the sections of solids and their coverings. Of these subjects, the first has already been explained in the series of Problems given in the geometrical part of this Work, and we are now about to treat on the other; that is, the Method of Covering them.

As no line can be formed on the edge of a single piece of timber, so as to arrange with a given surface, nor in the intersection of two surfaces, (by workmen called a *groin,*) without a complete understanding of both, the reader is required not to pass them until the operations are perfectly familiar to his mind. For the more effectually rivetting the principles upon the mind of the student, it is requested that he should *model* them as he proceeds, and apply the sections and coverings found on the paper to the real sections and surfaces, by bending them around the solid.

The Surfaces, which timbers are required to form, are those of *cylinders, cylindroids, cones, cuneoids, spheres, ellipsoids,* &c., either entire, or as terminated by cylinders, cylindroids, cones, and cuneoids.

The FORMATION of ARCHES, GROINS, NICHES, ANGLE-BRACKETS, LUNETTES, ROOFS, &c. depend entirely upon their *Sections*, or upon their *Covering*, or upon both.

This branch of carpentry, from its being subjected to geometrical rules, and described in schemes or diagrams upon a floor, sufficiently large for all the parts of the operation, has been called DESCRIPTIVE CARPENTRY.

In order to prepare the reader's mind for this subject, it will be necessary to point out the figures of the sections, as taken in certain positions.

ALL THE SECTIONS OF A CYLINDER, parallel to its base, are *circles*. All the sections of a cylinder, parallel to its axis, are *parallelograms*. And, if the axis of the cylinder be perpendicular to its base, all these parallelograms will be *rectangles*. If a cylinder be entirely cut through the curved surface, and if the section is not a circle, it is an *ellipse*.

ALL THE SECTIONS OF A CONE, parallel to its base, are *circles:* all the sections of a cone, passing through its vertex, are *triangles:* all the sections of a cone, which pass entirely through the curved surface, and which are not circles, are *ellipses*: all the sections of a cone, which are parallel to one of its sides, are denominated *parabolas;* and all the sections of a cone, which are parallel to any line within the solid, passing through the vertex, are denominated *hyperbolas*.

ALL THE SECTIONS OF A SPHERE OR GLOBE, made plane, are *circles*.

The solid formed by a SEMI-ELLIPSE, revolving upon one of its axes, is termed an *ellipsoid*.

ALL THE SECTIONS OF AN ELLIPSOID are similar figures: those sections, perpendicular to the fixed axis, are circles; and those parallel thereto are similar to the generating figure

OF THE COVERINGS OF SOLIDS.

PROBLEM 1.

To find the covering of the frustum of a right cone.

Let ABCD (*fig.* 1, *pl.* VII,) be the generating section of the frustum. On BC describe the semi-circle BEC, and produce the sides BA and CD, of the generating section ABCD, to meet each other in F. From the centre F, with the radius FA, describe the arc AH; and, from the same centre, F, with the radius FB, describe the arc BG; divide the arc, BEC, of the semi-circle, into any number of equal parts; the more, the greater truth will result from the operation; repeat the chord of one of these equal arcs upon the arc BG, as often as the arc BEC contains equal parts; then, through G, the extremity of the last part, draw GF, cutting the arc AH at H; then will ABGH be the covering required.

PROBLEM 2.

To find the covering of the frustum of a right cone, when cut by two concentric cylindric surfaces, perpendicular to the generating section.

Let ABCD (*fig.* 2, *pl.* VII,) be the given section, and AD, BC, the line on which the cylindric surface stands. Find the arc BG, as before, in *problem* 1, and mark the points, 1, 2, 3, &c. of division, both in the arc BG, and in the semi-circumference; from the points 1, 2, 3, &c. draw lines to F; also from the points 1, 2, 3, &c. in the semi-circumference, draw lines perpendicular to BC; so that each line thus drawn may meet or cut it. From the points of division in BC, draw more lines to F, cutting the arc BC in *a*, *b*, *c*, &c. From the points *a*, *b*, *c*, &c. draw lines parallel to BC, to cut the side BA: from the centre F, through each point of section in BA, describe an arc, cutting the lines drawn from each of the points 1, 2, 3, &c., in BG, at *a*, *b*, *c*, &c.; then will B*e*G be the curve, which will cover the line BC on the plan, or BC will be the seat of the line B*e*G.

In the same manner AH, the original of the line AD, will be found; and, consequently, B*e*GHA will form the covering over the given seat, ABCD, as required to be done.

PROBLEM 3.

To find the covering of a right cylinder.

Let ABCD (*fig.* 3, *pl.* VII,) be the seat or generating section. Produce the sides DA and CB to H and G, and on BC describe a semi-circle, and make the straight line BG equal to the semi-circumference: draw GH parallel to AB, and AH parallel to BG; then will ABGH be the covering required.

PROBLEM 4.

To find the covering of a right cylinder contained between two parallel planes, perpendicular to the generating section (*fig.* 4, *pl.* VII).

Through the point B draw IK, perpendicular to AB, and produce DC to K; on BK describe a semi-circle, and make BI equal to the length of the arc of the semi-circle, by dividing it into equal parts, and extending them on the line BI. Through the points of section, 1, 2, 3, &c., in the line BI, draw lines, 1*a*, 2*b*, 3*c*, &c., parallel to BA, and through the points 1, 2, 3, &c., in the arc of the semi-circle, draw the other lines 1*a*, 2*b*, 3*c*, &c., parallel to BA, cutting AD in *a*, *b*, *c*, &c. Draw *aa*, *bb*, *cc*, &c., parallel to BK; then, through the points, *a*, *b*, *c*, &c., draw the curve AH, and AH will be the edge of the covering over AD.

In the same manner the other opposite edge BG will be found, and the whole covering will therefore be ABGH.

PROBLEM 5.

ABCD (*fig.* 5, *pl.* VII,) being the seat of the covering of a semi-cylindric surface, contained between the surfaces of two other concentric cylinders, of which the axis is perpendicular to the given seat; it is required to find the covering.

Through B draw IK, perpendicular to AB; and produce DC to K. On BK describe a semi-circle, and divide its circumference into equal parts, at the points 1, 2, 3, &c.; the more of these the truer will be the operation; and repeat the chord on the straight line BI, as often as the arc contains equal parts, and mark the points 1, 2, 3, &c., of division. Through the points

1, 2, 3, &c., in the arc of the semi-circle, draw the lines 1*a*, 2*b*, 3*c*, &c., parallel to BA; and, through the points 1, 2, 3, &c., in BI, draw lines 1*a*, 2*b*, 3*c*, &c., parallel to BA. Draw *aa*, *bb*, *cc*, &c., parallel to KI, and through all the points *a*, *b*, *c*, &c., draw the curve line AH, which is one of the edges of the covering.

In the same manner the other edge BG will be found; and, consequently, the whole covering ABGH.

PROBLEM 6.

To find the covering of that portion of a semi-cylinder contained between two concentric surfaces of two other cylinders, the axis of these cylinders being perpendicular to ABCD (*fig.* 6, *pl.* VII).

Join BC, and, in this case, BC will be perpendicular to AB. Produce CB to G; and, on BC, describe a semi-circle. Divide the arc of the semi-circle into any number of equal parts, and extend the chords upon the straight line BG, marking the points of section both in the semi-circle and in the straight line BG. Through the points, 1, 2, 3, &c., in the arc of the semi-circle, draw lines 1*a*, 2*b*, 3*c*, &c., parallel to AB; and through the points 1, 2, 3, &c., in BG, draw the lines 1*a'*, 2*b'*, 3*c'*, &c., parallel to AB; also draw *aa'*, *bb'*, *cc'*, &c., parallel to BG, and, through the points *a*, *b*, *c*, &c, draw a curve, which will form one of the edges of the soffet; the opposite edge is formed in the same manner.

GROINS AND ARCHES.

Groins are the intersections of the surfaces of two arches crossing each other.

CONSTRUCTION OF GROINED ARCHES.

Groined Arches may be either formed of wood, and lathed over for plaster, or be constructed of brick or stone.

When constructed of brick or stone, they require to be supported upon wooden frames, boarded over, so as to form the convex surface, which each

vault is required to have, in order to sustain the cross arches during the time of turning them. This construction is called a CENTRE, and is removed when the work is finished. The framing consists of equidistant ribs, fixed in parallel planes, perpendicular to the axis of each body; so that, when the under sides of the boards are laid on the upper edges of the ribs, and fixed, the upper sides of the boards will form the surface required to build upon.

In the construction of the centering for groins, one portion of the centre must be completely formed to the surface of its corresponding vault, without any regard to the cross-arches, so that the upper sides of the boards will form a complete cylindric or cylindroidic surface. The ribs of the cross-vaults are then set at the same equal distances as that now described; and parts of ribs are fixed on the top of the boarding at the same distances, and boarded in, so as to intersect the other, and form the entire surface of the groin required.

Groins constructed of wood, in place of brick or stone, and lathed under the ribs, and the lath covered with plaster, are called *plaster-groins*.

PLASTER-GROINS are always constructed with diagonal ribs intersecting each other, then other ribs are fixed perpendicular to each axis, in vertical planes, at equal distances, with short portions of ribs upon the diagonal ribs; so that, when lathed over, the lath may be equally stiff to sustain the plaster.

When the axis and the surface of a semi-cylinder cuts those of another of greater diameter, the hollow surface of the lesser cylinder, as terminated by the greater cylinder, is called a *cylindro-cylindric arch*, and, vulgarly, a *Welsh groin*.

CYLINDRO-CYLINDRIC ARCHES, or *Welsh groins*, are constructed either of brick, stone, or wood. If constructed of brick or stone, they require to have centres, which are formed in the same manner as those for groins; and, if constructed of wood, lath, and plaster, the ribs must be formed to the surfaces.

In the construction of groins, and of cylindro-cylindric arches, the ribs that are shorter than the whole width are termed *jack-ribs*.

Cellars are frequently groined with brick or stone, and sometimes all the rooms of the basement stories of buildings, in order to render their superstructures proof against fire. The surfaces of brick or stone, on which the first arch stones, or course of bricks, are placed, are called the *springing of the arches.* It is evident that the more weight that is put on the side-walls, which sustain arches, the more will they be able to sustain the pressure of the arches; therefore the higher a wall is, the greater the weight will be on each of the side-walls; and for this reason groins are often constructed of wood in upper stories, instead of brick or stone, as not being liable to thrust out the walls, or bulge them, by the lateral pressure of the arches. The upper stories of buildings are never groined with stone or brick, unless when the walls are sufficiently thick to sustain the lateral pressure of the arches. The ceilings of Gothic buildings were frequently constructed with groined arches of stone, which were obliged to be supported with buttresses, at the springing points of the arches.

GROINS AND ARCHES.—PROBLEM 1. (*fig.* 1, *pl.* VIII.)

Given the plan of a rectangular groined arch or vault, of which the openings are of different widths, but of the same height, and a section of one of the arches, as also the seats of the groins, to find the covering of both arches, so as to meet their intersection.

In *fig.* 1, *pl.* VIII, let A, A, A, &c., be the plan of the piers, and *ab, cd,* the seats of the groins.*

Let the section of the arch, standing upon the lesser opening, BC, be a semi-circle: it is required to find the section upon the greater opening and the ends of the boards, so as to meet the groin, or line of intersection, of the two surfaces.

* The difference between the *plan* of any body and the *seat* of a point or line is distinguished thus: The *plan* is a figure upon which a solid is carried up, so that all sections, parallel to the plan, are equal and similar to that plan, and the surfaces are perpendicular; but the *seat* of a line is not in contact with the line itself; but a perpendicular erected from any point in the seat will pass through its corresponding point of the line itself.

On the diameter BC describe a semi-circle, and divide the quadrant into any number of equal parts, *ef*, *fg*, *gh*, &c., and from the points *e*, *f*, *g*, &c., draw lines, parallel to the axis F*k*, to meet the seat *ab* of the groined line, or line of intersection of the two surfaces. From the points *k*, *l*, *m*, &c. of intersection, draw the lines *k*Q, *l*R, *m*S, &c., parallel to the axis of the other vault, to meet the line VQ, perpendicular to that other axis in the points Q, R, S, &c. Then, upon any line, DE, transfer the points Q, R, S, &c. to *q*, *r*, *s*, &c., and draw *qv*, *rw*, *sx*, &c. perpendicular to DE, and transfer the ordinates F*e*, G*f*, H*g*, &c., of the semi-circle, to *qv*, *rw*, *sx*, &c., and through the points *v*, *w*, *x*, &c. draw a curve; then *qv*E will be half of the section required.

To find the covering of the semi-cylinder. Upon any straight line, YZ, No. 2, set off the distances *lm*, *mn*, *no*, &c., each equal to the chord *ef* or *fg*, &c., in No. 1; and draw *l*K, *m*L, *n*M, &c., in No. 2, perpendicular to YZ. Make *l*K, *m*L, *n*M, &c., No. 2, equal to L*k*, M*l*, N*m*, &c., of No. 1, and through the points K, L, M, &c., No. 2, draw a curve. Then will the figure K*l*Z be half of the covering of the cylinder.

To construct the covering, No. 3, for the great opening.

In the straight line *vq*, No. 3, make *vu*, *ut*, *ts*, &c., equal to the parts, E*z*, *zy*, *yx*, &c., of the elliptic curve, No. 1. In No. 3, draw *v*B, *u*O, *t*N, *s*M, &c., and make *v*B, *u*O, *t*N, *s*M, &c., No. 3, equal V*b*, U*o*, T*n*, S*m*, &c., No. 1; and in No. 3, draw a curve through the points B, O, N, M, &c.; then *qv*BK*q* will be the covering required.

GROINS AND ARCHES.—PROBLEM 2. (*fig.* 2, *pl.* VIII.)

To find the groin of a cylindro-cylindric arch.

Let A, A, A, A, be the plans of four piers, which form the openings of different widths. On the lesser opening PM, as a diameter, describe a semi-circle. Divide the quadrant next to P into any number of equal parts, and through the points of section draw the lines 1G, 2H, 3I, &c., perpendicular to PM, cutting PM in B, C, D, &c., and through the same points 1, 2, 3, &c., draw the lines 1*a*, 2*b*, 3*c*, &c., parallel to PM, cutting a line *qe* perpendicular to PM in the points *a*, *b*, *c*; produce the line which contains the points *a*, *b*, *c*,

through the greater opening; and upon the part of the line thus produced, which is intercepted between the piers A, A, describe a semi-circle. Produce the line MP to *k*, and from *q* describe arcs *af*, *bg*, *ch*, &c., cutting B*k* in the points *f*, *g*, *h*, &c. Draw *fk*, *gl*, *hm*, &c., parallel to the base of the greater semi-circle, to cut the arc of the same in the points, *k*, *l*, *m*, &c. From the points *k*, *l*, *m*, &c., draw the lines *k*G, *l*H, *m*I, &c., parallel to PM; then, through the points G, H, I, K, L, draw a curve GHIKL, which will be the seat of the groin.

The covering to coincide with the groin is shown at No. 1. Draw *pm*, No. 1, and make *pb*, *bc*, *cd*, &c., each equal to P1; 1, 2; 2, 3, &c., in the semi-circular arc. In No. 1, draw *pq*, *bg*, *ch*, &c., respectively equal to BG, CH, DI, &c., and through the points *q*, *g*, *h*, *i*, &c., draw a curve; then will *pqnm* be the covering required.

GROINS AND ARCHES.—PROBLEM 3. (*fig.* 3, *pl.* VIII.)

To find the diagonal or groin-rib of a VAULT, of which the lesser openings are semi-circles, and the groins, in vertical planes, passing through the diagonals of the piers.

On *ah*, (*fig.* 3, *pl.* VIII,) the perpendicular distance between two adjacent piers of the lesser opening, describe a semi-circle, *abh*; and, in the arc, take 1, 2, 3, &c., any number of points, and draw the lines 1*l*, 2*m*, 3*n*, &c., cutting the diagonal *ik*, in *l*, *m*, *n*, &c. Draw, as before, *lq*, *mr*, *ns*, &c., perpendicular to *ik*, and through the points *i*, *q*, *r*, *s*, &c. draw a curve; then *iuk* will be the edge of the rib to be placed in the groin.

The edge of the rib, for the other opening, will be found thus: From the points *l*, *m*, *n*, &c., draw the lines, *l*I, *m*K, *n*L, &c., parallel to the axis of the opening of the larger body, cutting HB at the points C, D, E, &c. Make CI DK, EL, &c., each equal to *c*1, *d*2, *e*3, &c.; then, through the points B, I, K, L, &c., draw a curve; and the line thus drawn will be in the surface of the greater opening, so that BNH will be one of the ribs of the body-range.

The method of placing the ribs is exhibited at the lower end of the diagram, *fig.* 3, the ribs of each opening being placed perpendicular to the axis of each groin.

GROINS AND ARCHES.—PROBLEM 4. (*fig.* 4, *pl.* VIII.)

To find the groined and side ribs of a LUNETTE, where the groined ribs are in vertical planes upon the straight lines *ag*, *gl*, (*fig.* 4, *pl.* VIII,) the principal arch being a semi-circle.

Let AC be the base of one of the principal arches, perpendicular to one of the sides of the main vault, the points A and C being in the same range with those sides. Let *mq* be the opening of one of the lunette windows. From the point *g*, the meeting of the two seats of each groin, draw *gr* perpendicular to *mq*, cutting *mq* at *n*; draw *g*3 parallel to *mq*, cutting the semi-circular arc ABC at 3. Between A and 3 take any number of intermediate points, 1, 2, &c., and, through the points 1, 2, &c., thus assumed, draw 1*e*, 2*f*, &c., cutting the seat *ag*, of the first groin, in the points *e*, *f*, &c., and AC in *b*, *c*, *d*, &c. Perpendicular to *ag* draw *eh*, *fi*, &c., and make *eh*, *fi*, *gk*, each equal to *b*1, *c*2, *d*3, &c.; then, through the points *g*, *h*, *i*, *k*, draw a curve, which will form the groin belonging to the seat *ag*. From the points *e*, *f*, &c., draw lines *et*, *fs*, &c., cutting *qm* in the points *p*, *o*, &c.; and, through the points *q*, *t*, *s*, *r*, draw the curve *qtsr*, which will be one of the ribs of the lunette.

GROINS AND ARCHES.—PROBLEM 5. (*fig.* 5, *pl.* VIII.)

Given one of the ribs of a LUNETTE, and a rib of the main arch, to determine the seat of the groin, or the seat of the intersection of the two surfaces.

This is, in fact, a cylindro-cylindric arch; we shall therefore refer the reader to *Problem* 2, for the geometrical construction of the same.

LUNETTES are used in large rooms or halls, and are made either in waggon-headed ceilings, or through large coves, surrounding a plane ceiling: they have a very elegant effect when they are numerous, and disposed at equal distances. Though it is not necessary to have the axes of the lunettes and the axes of the quadrantal cylindric surfaces in the same plane, they have the best effect when executed so; as the groin, formed by the meeting of the two surfaces, has, in this case, less projection: and, though the groins are curves of double curvature, their seats are perfect hyperbolas, and may be

described independent of the rules of projection, the summit or vertex of the curve being once ascertained: by these means we shall have the abscissa and double ordinate; the transverse axis being the distance between the opposite curves.

GROINS AND ARCHES.—PROBLEM 6. (*fig.* 1, *pl.* IX.)

To find the groin of a CYLINDRO-CYLINDRIC ARCH, and the moulds for the boarding.

A Cylindro-cylindric Arch is the intersection of one semi-cylinder, of a less diameter, with another of a greater diameter. The principal objects to be found are, the seat of the curve on the plan, and the moulds for terminating the ends of the boards.

For this purpose, on any straight line, which has A at one of its ends, as a diameter, describe a semi-circle, as at No. 1, in the figure, terminating in A, for the section of the greater vault, or semi-cylindric arch. As the axis of the one cylinder is supposed to cut the axis of the other at right angles, the sides of the cross-vaults will also be at right angles to each other: therefore draw the diameter AC, of the lesser vault, perpendicular to the diameter of the greater vault; and on AC, as a diameter, describe the semi-circle ABC: divide the quadrantal arc AB into any number of equal parts, as here into five. Draw A*e* perpendicular to AC, and produce CA to *k*. Through the points of division, in the quadrantal arc AB, draw 1*a*, 2*b*, 3*c*, 4*d*, B*e*, cutting A*e*, in *a*, *b*, *c*, *d*, *e*. Again, through the same points 1, 2, 3, 4, B, in the quadrantal arc AB, draw straight lines 1*q*, 2*r*, 3*s*, 4*t*, BD, perpendicular to AC. From the point A, as a centre, with the several distances A*a*, A*b*, A*c*, A*d*, A*e*, describe the arcs *ek*, *di*, *ch*, *bg*, *af*, cutting A*k* in *f*, *g*, *h*, *i*, *k*.

Parallel to the diameter of the greater semi-circle, or parallel to A*e*, (*fig.* 1, No. 1,) draw *fl*, *gm*, *hn*, *io*, *kp*, cutting the greater semi-circular arc in the points *l*, *m*, *n*, *o*, *p*. Through the points *l*, *m*, *n*, *o*, *p*, draw *lq*, *mr*, *ns*, *ot*, *p*D, parallel to AC, cutting the perpendiculars 1*q*, 2*r*, 3*s*, 4*t*, BD, in the points *q*, *r*, *s*, *t*, D. Through the points A, *q*, *r*, *s*, *t*, D, trace a curve by hand, or put in nails at the points A, *q*, *r*, *s*, *t*, D, and bend a thin slip of wood so as to come in

contact with all the nails; then, by the edge of this slip, which touches the nails, draw a line with a pencil, or find points; and the curve thus drawn will be half the seat of the rib. The other half, being exactly the reverse, may be found by placing the distances of the ordinates at the same distance from the centre, upon the diameter AC, and setting up the perpendiculars by making them respectively equal to the others.

It will perhaps be eligible to make the whole curve ADC at once.

The mould for cutting the ends of the boards, which are to cover the centres of the lesser openings, will be found as follows:

On any straight line, C5, as on the diameter AC produced, set off the equal parts A1; 2,3; 3,4; 4B; of the quadrant AB, on the straight line C5, from C to 1, from 1 to 2, from 2 to 3, from 3 to 4, from 4 to 5, and draw the straight lines 1*u*, 2*v*, 3*w*, 4*x*, 5*y*, perpendicular to C5. Make 1*u*, 2*v*, 3*w*, 4*x*, 5*y*, each respectively equal to each of the ordinates comprehended between the base AC, and the seat AD; then, through all the points C, *u*, *v*, *w*, *x*, *y*, draw a curve C*uvwxy*, as before; then the shadowed part, of which the curve line C*uvwxy* is the edge, is the *mould* for one side, which may also be made use of for the other.

To apply this mould, all the boards should be laid together, edge to edge, on a flat or plane surface, to the breadth C5. Draw a straight line C5, perpendicular to the edge of the first board, at the distance of 5*y* from the end. At the distance C5 draw a perpendicular 5*y*, and set off the distance 5*y*. Then apply the proper edge of the mould from C to *y*, as exhibited in the plate, and draw a curve across the boards, and cut their ends off by the line thus drawn; then the ends, thus formed of the remaining parts, will fit upon the boarding of the greater vault, after being properly bevelled, so as to fit upon the surface of the said boarding.

No. 4, of *fig.* 1, exhibits the curve, in order to draw or discover the line on the boarding of the greater vault, in order to place the boarding of the lesser vault.

Nos. 2 and 3, *fig.* 1, show the method of forming the inner edges of the ribs, so as to range with the small opening. The under edge of the rib must be

formed so as to correspond to the curve which is its seat; and the little distances, between the straight line and the curve, must be set off on the short lines, shown at Nos. 1, 2, and 3; then a curve may be drawn through the points of extension, and the superfluous wood taken away; then, the rib being put in its real place, the angle will exactly fall over its seat. The diagram, *figure* 1, and its different numbers, answer both the purposes of a centering and of ribbing for plaster-ceilings.

Figure 2, *pl.* IX, exhibits the method of forming the *Cradelling*, or ribs, for plaster-ceilings of cylindro-cylindric arches. Here principal ribs only are used across the piers. The ribs of double curvature, which form the groins, though here exhibited, in order to fix the ribs, may be done without, by men of experience; but young workmen require every assistance, in order to acquire a comprehensive idea of the subject; it is, therefore, proper to show how the groined ribs are to be found. The other ribs, for lathing upon, are made of straight pieces of quartering, fixed equidistantly.

Figure 3, *pl.* IX, is a plan in which common groins and cylindro-cylindric arches both occur. See the gate-way leading from the Strand, in London, into the court of Somerset-house.

GROINS AND ARCHES.—PROBLEM 7. (*fig.* 4, *pl.* IX.)

To find the seats of the intersections of groins formed by the intersection of an annular and a radial vault, both being at the same height, the section of the annular vault being a semi-circle, and that of the radiating vault a semicircle of the same dimensions, the plan being given.

Perpendicular to the middle line, or axis, of the radial vault, draw a straight line from any point of that middle line; from the point thus drawn, set the radius of the circle of the annular vault; from the point of extension draw a line, parallel to the axis of the radiating vault, to meet the side of the plan. From the point of meeting draw a straight line, perpendicular to the axis, to meet the other side of the plan of that radiating vault: on the perpendicular thus drawn, between the two sides, as a diameter, describe a semi-circle: divide each quadrantal arc of this semi-circle, and each quadrantal arc of the

semi-circle which is the section of the annular vault, into the same number of equal parts. Draw lines through the points of division in each arc, perpendicular to the base or diameter, to meet the said diameter. Through the points of section in the diameter of the annular vault, and from the point of concourse of the two sides of the radiating vault, describe arcs. From the same point of concourse, and through the points of section of the diameter of the semi-circle, which is the section of the radiating vault, draw lines from the point of concourse of the two sides of the radiating vault. Then, through the intersection of these lines, and the arcs drawn from the points of section in the diameter of the semi-circle, which is the section of the annular vault, trace a curve, which will be the seat of the groin. The method of fixing the timber is exhibited at the other end of the figure. The ribs of both the annular vault and the radiating vault are all fixed in right sections of these vaults, as must appear evident from what has been shown.

NAKED FLOORING.

FLOORS are those partitions in houses that divide one story from another.

FLOORS are executed in various ways: some are supported by single pieces of timber, upon which boards for walking upon are nailed. Floors of this simple construction are called *single-joisted floors*, or *single floors;* the pieces of timber, which support the boards, being called *joists*. It is, however, customary to call every piece of timber, under the boarding of a floor, used either for supporting the boards or ceiling, by the name of *joists*, excepting large beams of timber into which the smaller timbers are framed.

When the supporting timbers of a floor are formed by one row laid upon another, the upper row are called *bridging joists*, and the lower row are called *binding joists*. Sometimes a row of timbers is fixed into the binding joists, either by mortises and tenons, or by placing them underneath, and nailing them up to the binding-joists: these timbers are called *ceiling-joists*, and are used for the purpose of lathing upon, in order to sustain the plaster-ceiling.

In forming the naked flooring, over rooms of very large dimensions, it is found necessary to introduce large strong timbers, in order to shorten the bearing of the binding-joists; such strong timbers are called *girders*, and are made with mortises, in order to receive the tenons at the ends of the binding-joists, which, by this mean, are greatly stiffened, being much shorter.

The *bridging-joists* are frequently notched down on the binding-joists, in order to render the whole work more steady.

Figure 1, *pl.* X, is the plan of a naked floor; *b, b, b,* &c. are the binding-joists; *a, a, a,* &c. are the bridging-joists; *d,* a timber close upon the stair-case. This piece of timber is called a *trimmer:* its use is to receive and secure the ends of the joists, *e, e, e,* &c. upon the landing.

C, C, C, &c. are *wall-plates,* upon which the ends of the binding-joists rest.

In the construction of floors, great care must be taken that no timber come near to a chimney; therefore, the ends of the timbers, as here shown, have no connection with the fire-place, nor with the flues.

The flues, in this plan, are indicated by their being shadowed darker than the other parts of the plan.

Figure 2, *pl.* X, is the plan of naked flooring with a girder.

Figure 3, shows the manner of framing joists into a girder, with the form of the mortise and tenon. No. 1, is the part of a joist framed into the girder; and No. 2 is a joist out of the mortise.

Figure 4, shows the connection of binding-joists, bridging-joists, and ceiling-joists; as, also, the manner of fixing the binding-joists upon the wall-plates, which manner is called *cocking*, or *cogging*. The long dark parts represent the mortises, into which one end of the ceiling-joists are fixed. These long mortises are called *pulley-mortises,* or *chase-mortises.* The ceiling-joists are introduced into common mortises at one end, and the other end of them are let into these long mortises obliquely, and slide along until they are perpendicular.

Figure 5, shows how the bridging-joist is let down upon the binding-joist, and how the ceiling-joists are fitted into the binding-joists.

TUMBLING IN A JOIST, is to frame a joist between two timbers, of which the sides, which ought to be vertical or square to the upper edges, are oblique to these edges.

Figure 6, shows the method of fitting in a joist between the sloping sides of two others. The first thing done is, to turn the upper edge of the joist upon the top of the two pieces into which it is to be fitted, and brought over its proper place. The next thing is to turn the joist on its under edge, so as to lie over its place; then apply a rule, or straight edge, upon the side of the one piece where the shoulder of the joist is intended to come; then slide the joist until the line drawn come to the straight edge of the rule so applied; then draw a line by the edge of the rule. Do the same at the other end, and the two lines thus drawn will mark the shoulder of the tenon at each end.

LENGTHENING TIMBERS.

TIMBERS may be lengthened in various ways, either by making the piece of timber in two or more thicknesses; or by securing one piece to another, with a piece on each side, in order to cover the joint; and by spiking or bolting each piece on both sides of the joint. Sometimes the pieces that are applied on the sides are made of wood; in this case, it is called *fishing the beam:* such modes are used in ships, when their masts, beams, or yards, are broken, in order to mend them. Other modes of continuing the length of timbers or beams is, by splicing them with a long bevel-joint, ending in a sharp edge at the end of each piece. Sometimes the sharp edge of the end of each piece is cut off, so as to form an obtuse angle at the top. Sometimes the splice is so formed as that the two surfaces which come into contact are reciprocally indented into each other, which will add greatly to their security, when firmly bolted together. Every kind of scarf should have a strong iron-strap upon each opposite side, extending in length considerably beyond each joint.

Figure 7, *pl.* X, shows the manner of building a beam in three thicknesses; which, being strapped with iron across every joint, and bolted, will be exceedingly strong and firm.

Figure 8, exhibits the method of joining timbers by two *tables* and a *key*.

Figure 9, the method of lengthening timber by a plain scarf, being cut only with an obtuse angle at the ends.

Figure 10, the same kind of scarf, with two tables and a key.

Timbers that are scarfed and strapped ought to be so applied, that the sides which are strapped should be the horizontal sides; for, if otherwise applied, they will be liable to split at the bolting.

But, if the surfaces of the joints are to be placed in a vertical position, there ought to be two straps upon the top and two upon the bottom; each strap being brought close to the vertical face. By this method it will be much stronger than when set in the other position, or with the joint of the scarf horizontal.

THE ROOFING.

THE ROOF is that part of a building raised upon the walls, and extending over all the parts of the interior, in order to protect its contents from depredation, and from the severities and changes of the weather.

The ROOF, in CARPENTRY, consists of the timber-work which is found necessary for the support of the external covering.

The most simple form of roofs is that consisting of a level plane; but this description of roofs is adapted only to short bearings, and is not at all calculated to resist or prevent the torrents of rain or moisture from penetrating into the interior.

The next simple form is that which consists of an inclined plane; and, though well calculated to resist the injuries of the weather, and to afford greater strength than a level disposition of the timbers would supply, it is far from admitting of the utmost strength that a given quantity of timber is

capable of affording: it occasions an inequality, and a want of uniformity and correspondence in the proportions of the fabric, and an unnecessary and unpleasant height of walling. The best figure for a roof is that which consists of two equal sides, equally inclined to the horizon, terminating in the summit, over the middle of the edifice, in a horizontal line, or the *ridge* of the roof, as it is called: so that the section made by a plane, perpendicular to the ridge, is every where an isosceles triangle, the vertical angle of which is the top of the roof. This form is very advantageous, as it regards saving of timber; for it may be executed with the same scantlings, to span double the distance, that the simple sloping roof admits; or, in buildings of the same dimensions, the scantlings of the timbers will be very much diminished.

The antient Egyptians, Babylonians, and other eastern nations, of the remotest antiquity, constructed their roofs flat, as do likewise the present inhabitants of these countries. The antient Greeks, though favoured with a mild climate, yet sometimes liable to rain, soon found the inconvenience of a platform covering for their houses, and accordingly raised the roof in the middle, declining towards each side of the building, by a gentle inclination to the horizon, forming an angle of from 13 to 15 degrees, or the perpendicular height of from one-eighth to one-ninth of the span.

In Italy, where the climate is still more liable to rain, the antient Romans constructed their roofs from one-fifth to two-ninth parts of the span.

In Germany, where the severities of the climate are still more intense than in Italy, the antient inhabitants, as we are informed by *Vitruvius*, made their roofs of a very high pitch. When the pointed style of architecture was introduced into Europe, high pitched roofs were thought consonant with its principles; and they therefore formed, externally, one of the most striking characteristics of the Gothic style.

In their usual proportions, the rafters were equal to the breadth or span of the roof, or the rafters were the sides of an equilateral triangle, of which the spanning line was the base.

During the middle ages this form prevailed, not only in public but in private buildings, from the most stately and sumptuous mansion down to

the humble cottage of the common labourer; and this equilateral triangular roof continued in request until, finally, the pointed style came into disuse.

When the celebrated INIGO JONES introduced Roman architecture, the rafters were made three-quarters of the breadth of the building; and this proportion, which was called *true pitch*, still prevails in some parts of the country where plain tiles are used; subsequently, also, the square seems to have been considered as *true pitch:* but, in large mansions, constructed in the Italian style, roofs of a *pediment pitch*, covered with lead, were introduced.

In the present day, where good slates are to be obtained in abundance, roofs may be covered with them, from the pyramidal equilateral Gothic down to the gently inclined Greek pediment.

With regard to the present practice, the proportion of the roof depends on the style of the architecture of the edifice; the usual height varying from one-third to one-fourth part of the span.

There are, doubtless, some advantages in high pitched roofs, as they discharge the rain with greater rapidity; the snow does not lodge so long on their surface; also, they may be covered with smaller slates, and even with less care, and are not so liable to be stripped by high winds as the low roofs are; but the low roofs bear less weighty on the walls, and are considerably cheaper, since they require shorter timbers, and, of course, smaller scantlings.

The roof is one of the principal ties to a building, when executed with judgement, as it connects the exterior walls, and binds them together as one mass; and, besides the protection it affords the inhabitant within, it preserves the whole work from a state of decay, which would soon inevitably ensue, from the violence of rain or frost, which would operate in a way of rotting the timbers, of destroying the connection in the walls, and would cause them ultimately to fall.

The several timbers of a roof are, *principal rafters, tie-beams, king-posts, queen-posts, struts, collar-beams, straining-sils, pole-plates, purlins, ridge-piece, common-rafters*, and *camber-beams*. The uses of these will appear from the following description of them.

The usual EXTERNAL FORM of a ROOF has two surfaces, which generally rise from opposite walls, with the same inclination; and, as the walls are most commonly built parallel to each other, the section of the roof made by a plane perpendicular to the horizon, and to one of the walls, is an isosceles triangle; the base being the extension from the one wall head to the other. This extension is called the *span of the roof.*

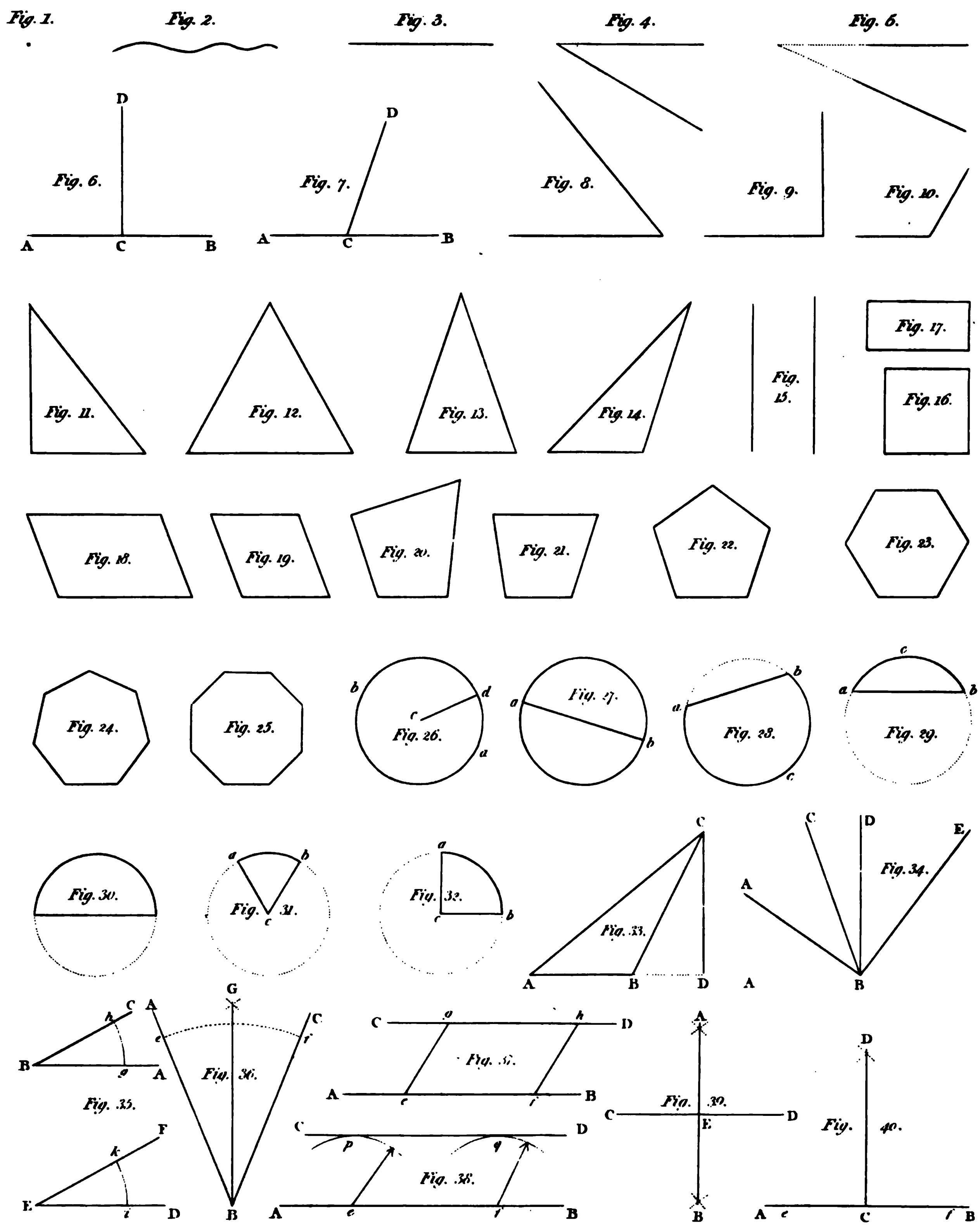
To FRAME TIMBERS, so that their external surfaces shall keep this position, is the business of *trussing;* and the ingenuity of the carpenter is displayed in making the strongest roof with a given quantity of timbers.

All beams, or pieces of timber, from their weight, when supported at the two ends only, acquire a concavity on the upper side; and this concavity is the greater as the distance between the props is the greater. It is, therefore, a grand object to prevent this bending as much as possible. The curvature will take place whether the position of beams be horizontal or inclined; but the same beam will have less curvature, as the angle, to which it is inclined to the horizon, is greater. For, it is evident that, when a beam is laid level, and supported at its extremities, its curvature will be greater than when inclined at any angle, however small; and, again, if it stand perpendicular to the horizon, its curvature will be nothing; that is to say, its curvature will be nothing when the angle of inclination is the greatest.

The curvature which timber obtains by bending is called *sagging*. To prevent timber from sagging, as much as is possible, it must be supported at a certain number of intermediate points or places, besides the two extreme points or places. Now these supports must themselves be supported from some base or other; but, if the resting points or places is upon the surface or surfaces of other timbers, the greatest care must be taken that they do not fall between the extremities of the supporting timbers intended to support the other. That is to say, the lower end of every piece of timber, used as a prop, must rest upon some fixed point; or, otherwise, the propping piece of timber must be so disposed that the pressing forces at each end must be equal to each other.

These are the general principles upon which the strength of roofs depend.

GEOMETRY.

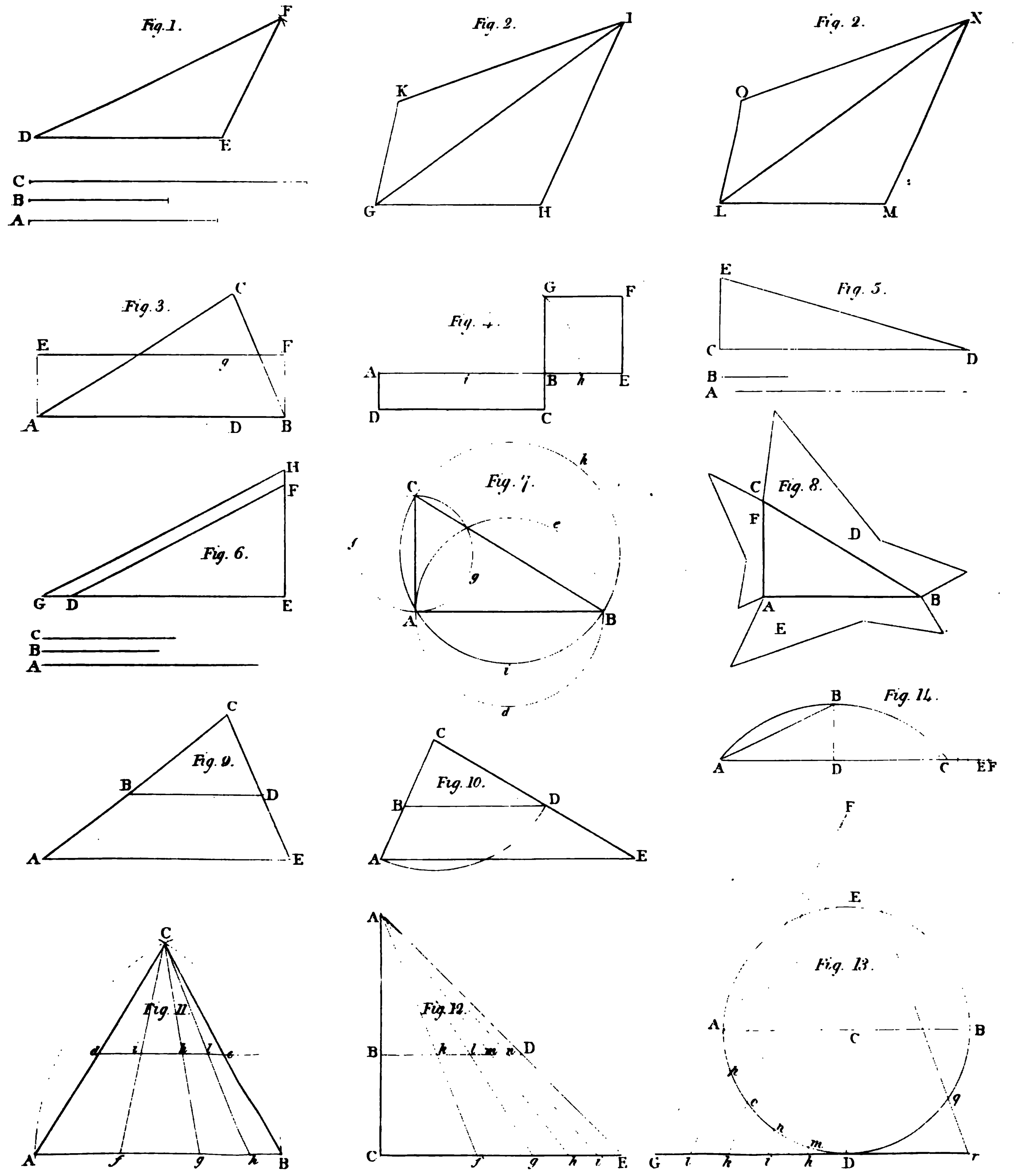


Drawn by P. Nicholson.

Engraved by W. Symns.

London, Published by Thos. Kelly, 17 Paternoster Row, January 1 18[illegible].

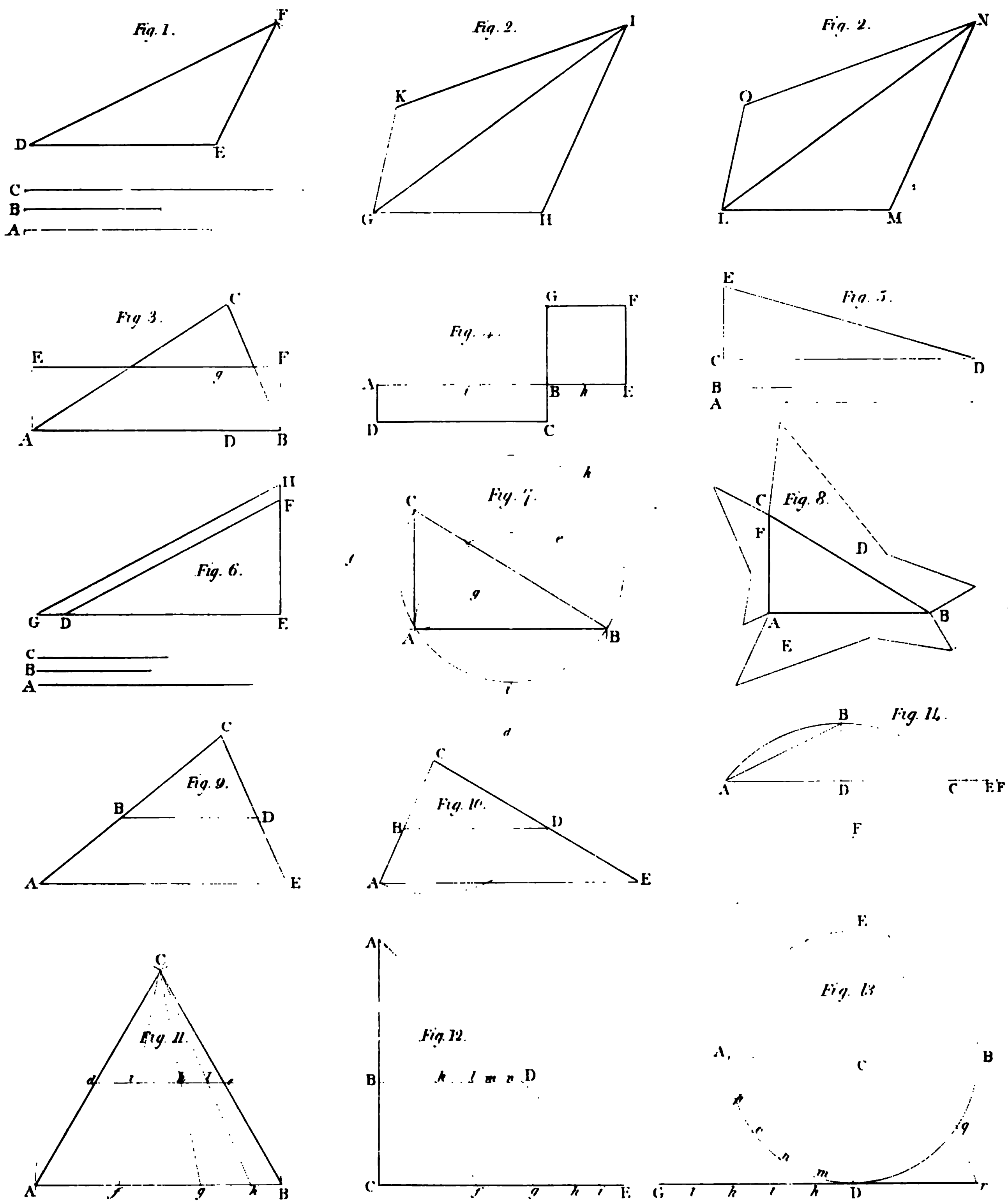
GEOMETRY.



Drawn by P. Nicholson. *Engraved by W. Symns.*

London. Published by Thos. Kelly. 17, Paternoster Row. Sepr. 16. 1823.

GEOMETRY.



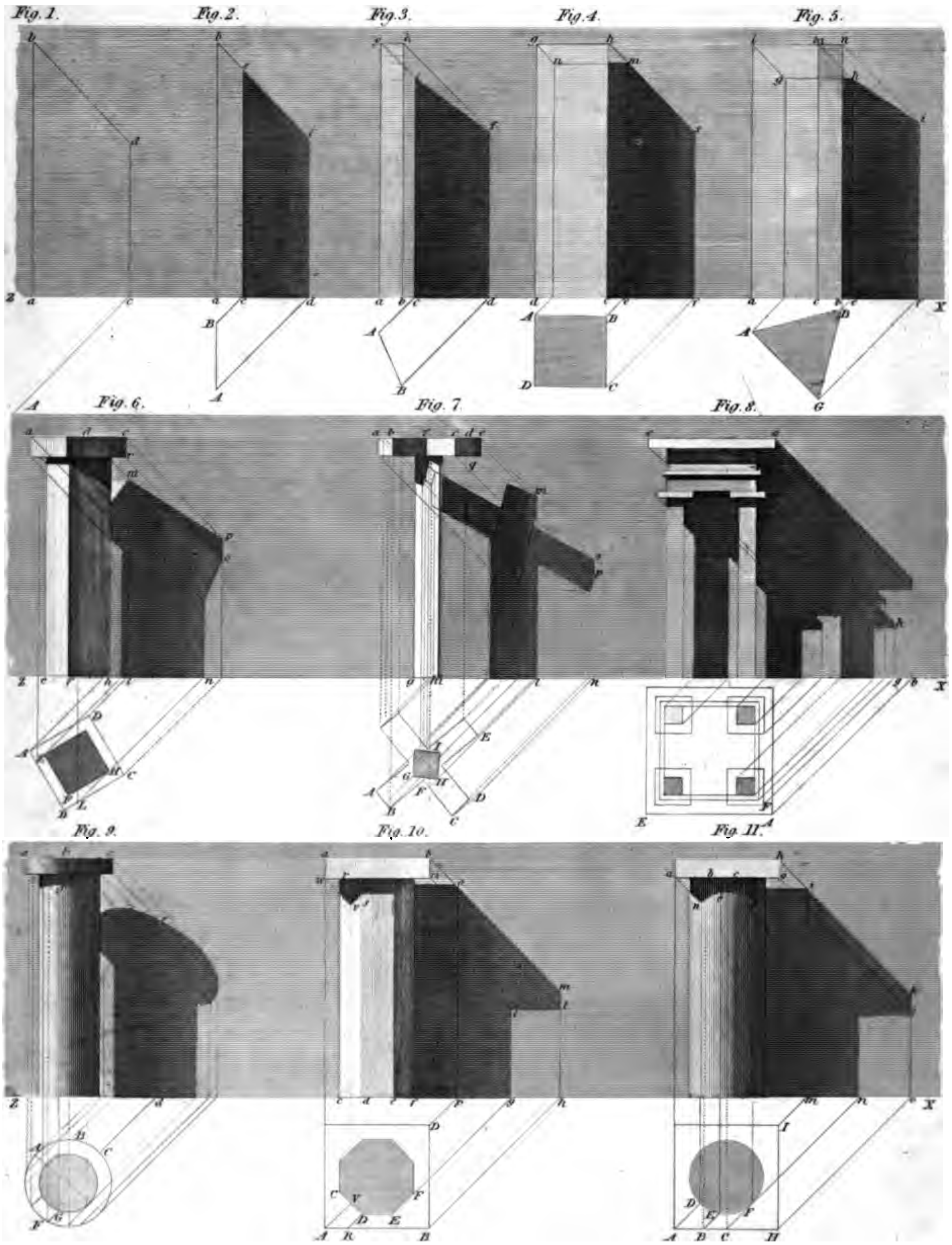
Drawn by P. Nicholson. Engraved by W. Symns.

London. Published by Thos. Kelly. 17, Paternoster Row. Sept. 16. 1823.

PROJECTION.

PLATE IV.

SHADOWS.



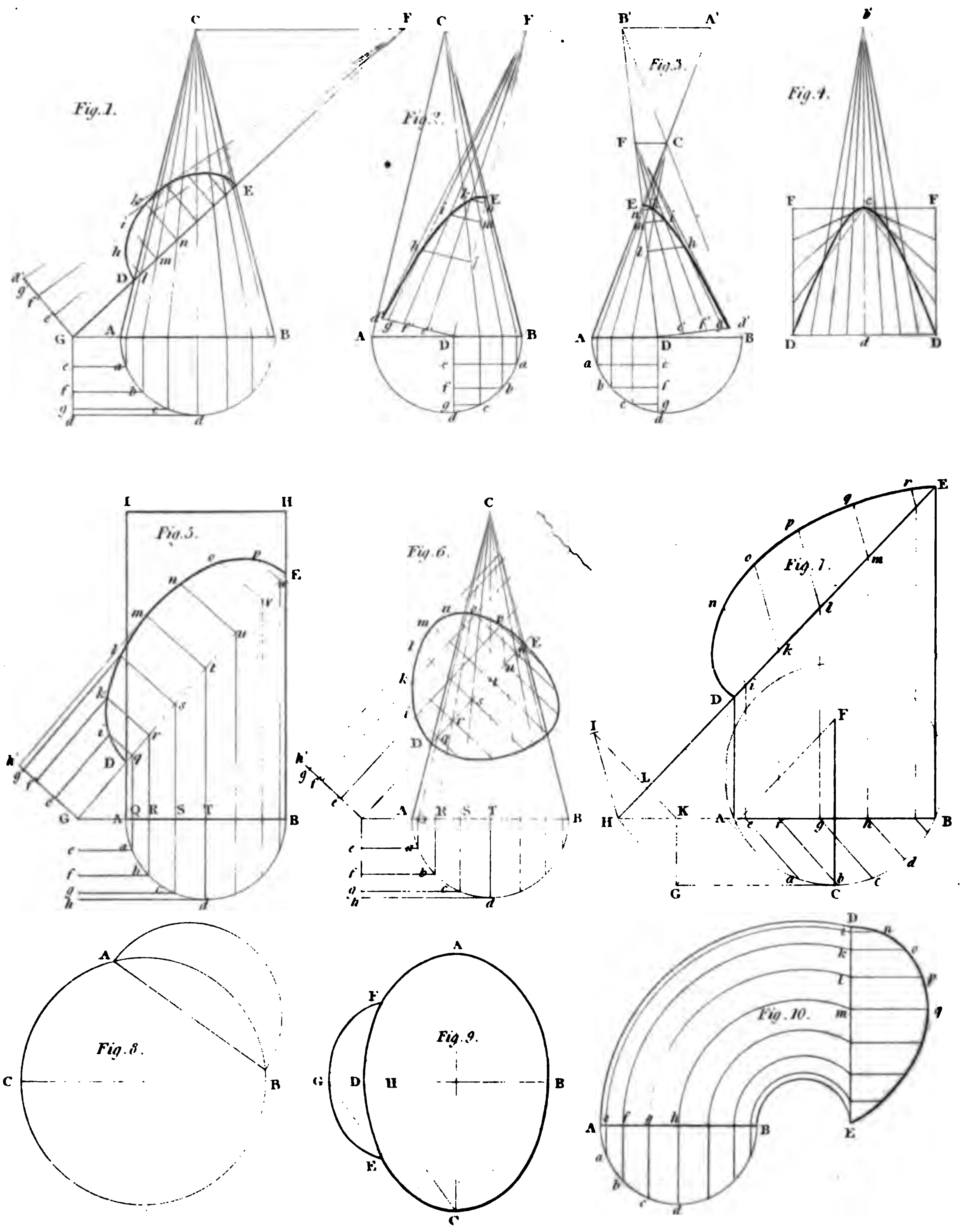
M. A. Nicholson.

Engraved by

London, Published by Thos Kelly, 17, Paternoster Row, Sepr 5, 1825.

GEOMETRY.

SECTIONS OF SOLIDS.



Drawn by P. Nicholson.

Engraved by W. Symns.

London, Published by Thos. Kelly, 17 Paternoster Row, January 1.18[illegible]

GEOMETRICAL LINES.

FOR ROOFS.

Fig. 1.

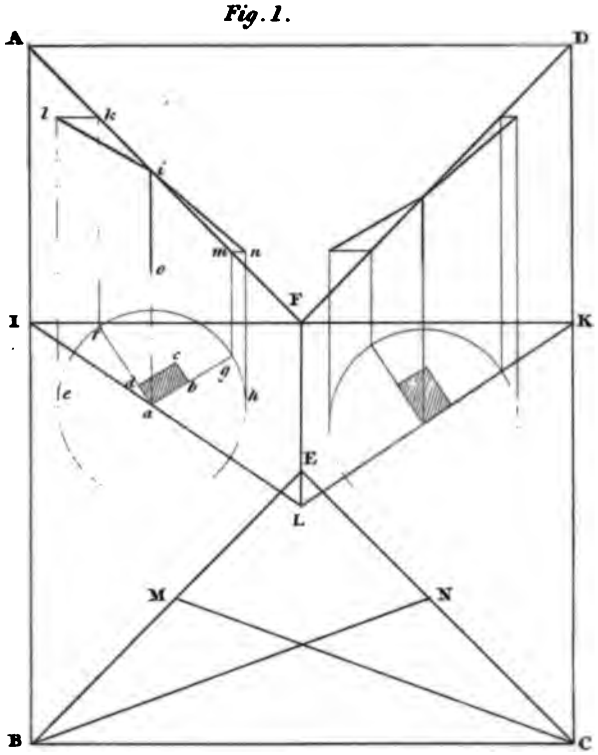


Fig. 2.

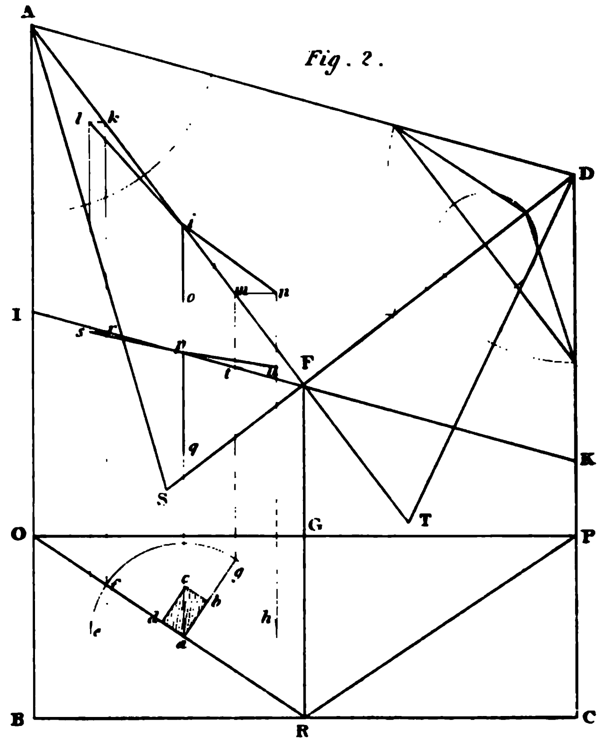


Fig. 3.

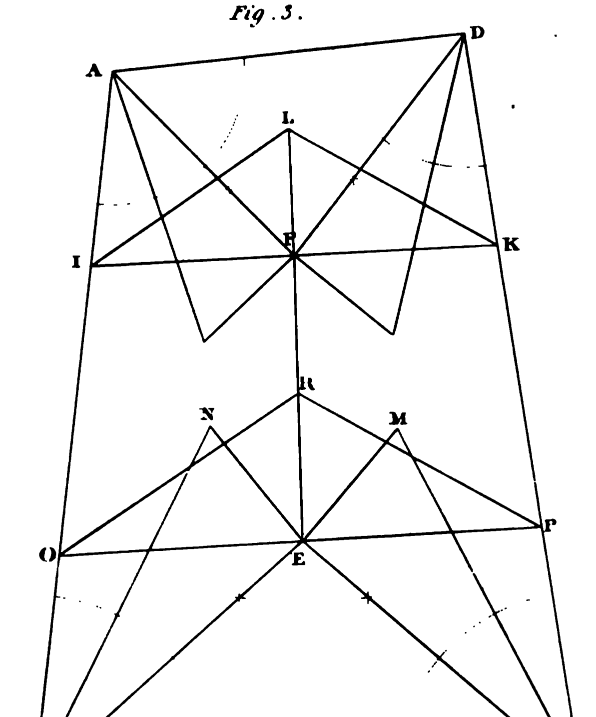
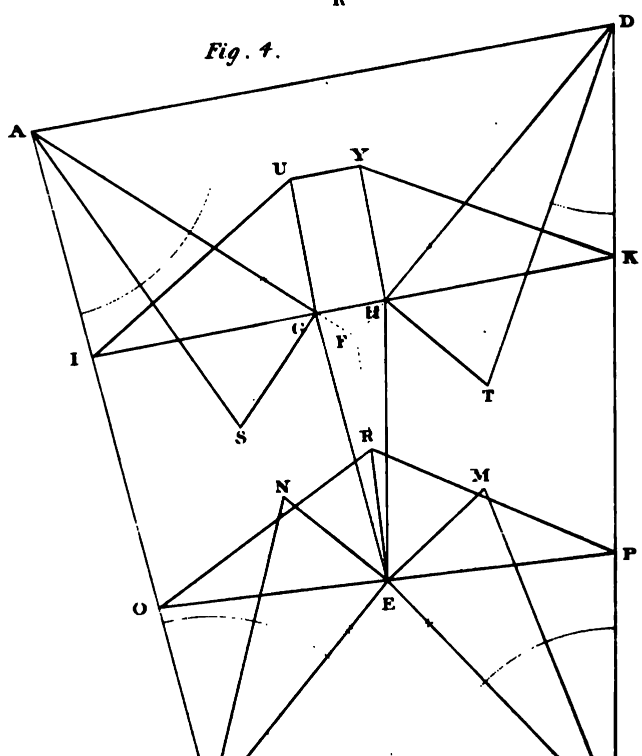


Fig. 4.

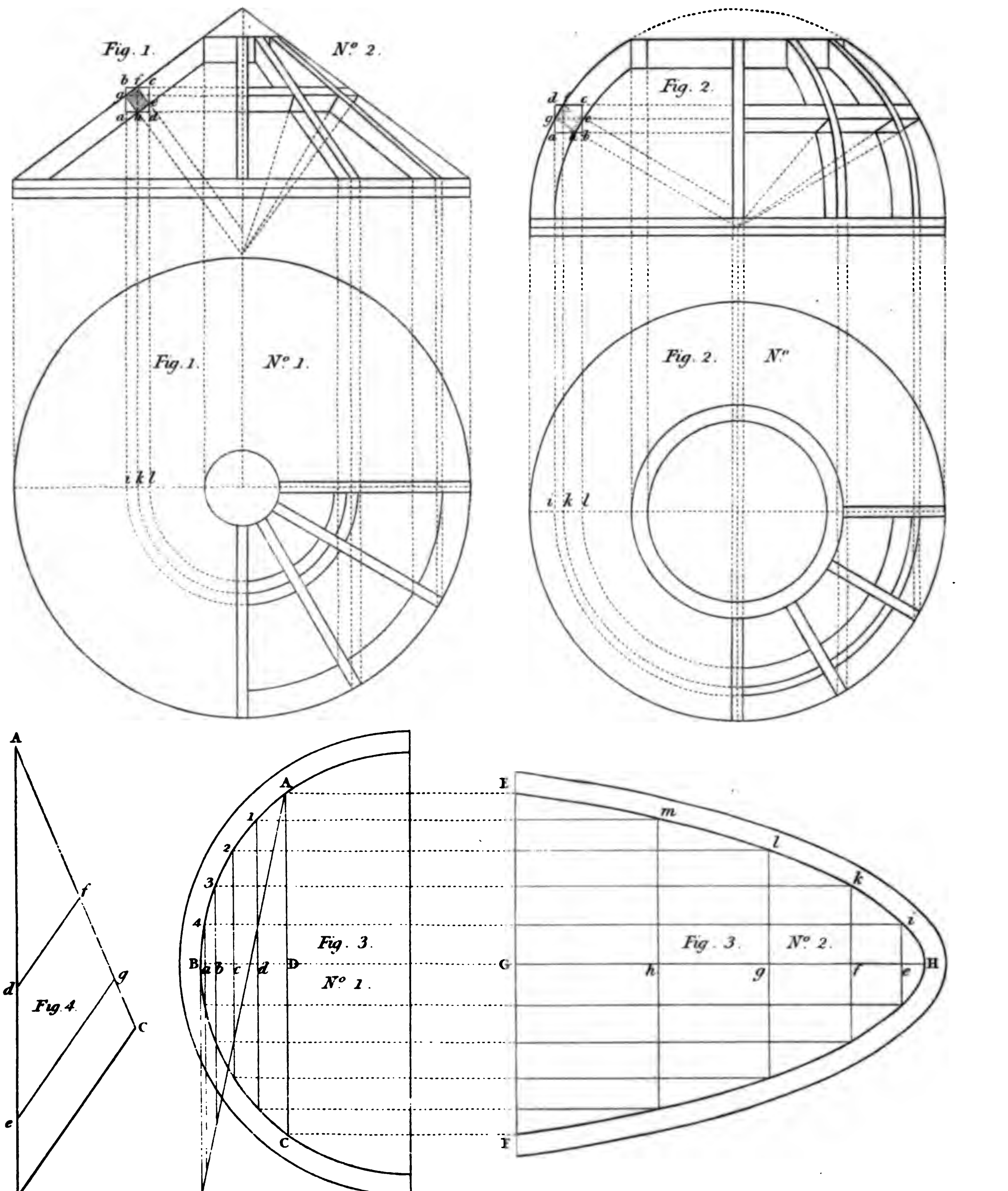


...wn by P. Nicholson.

Engraved by W. Symms

London, Published by Thos Kelly, 17 Paternoster Row, January 1 18..

PURLINS IN CIRCULAR ROOFS.



Drawn by P. Nicholson.

Engraved by W. Symns.

London, Published by Thos. Kelly, 17, Paternoster Row, Jany. 1, 1822.

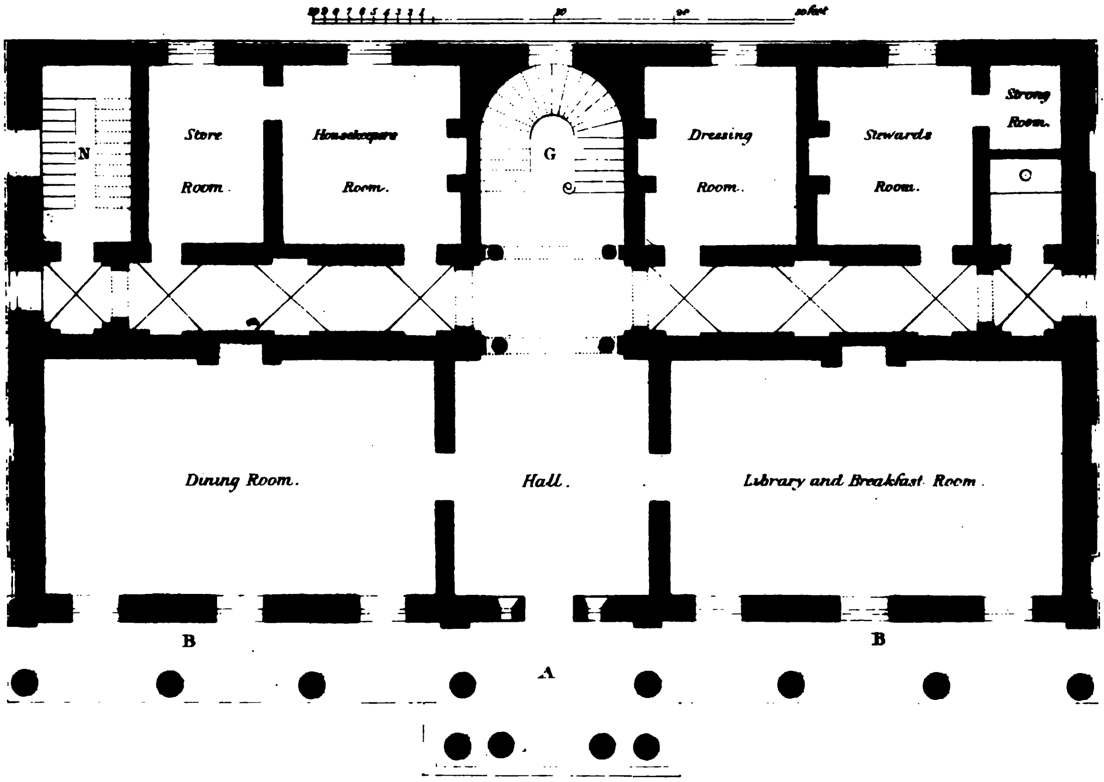


Fig. 2.

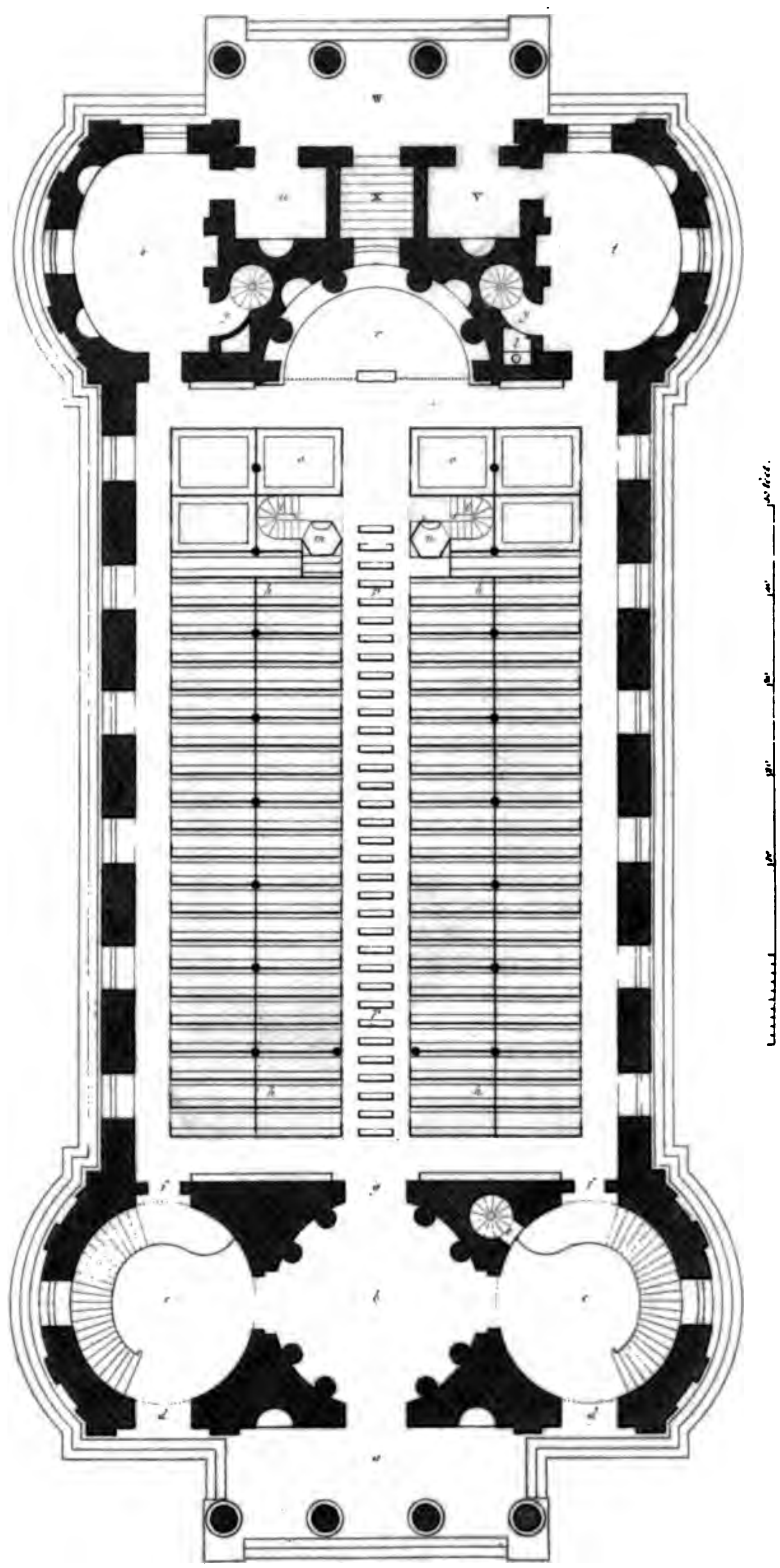
GROUND PLAN.

Designed by M.A. Nicholson

Engraved by W. Symns.

London. Published by Thos. Kelly 17 Paternoster Row July 5 1823.

GROUND PLAN, OF A CHURCH IN THE GRECIAN STYLE.



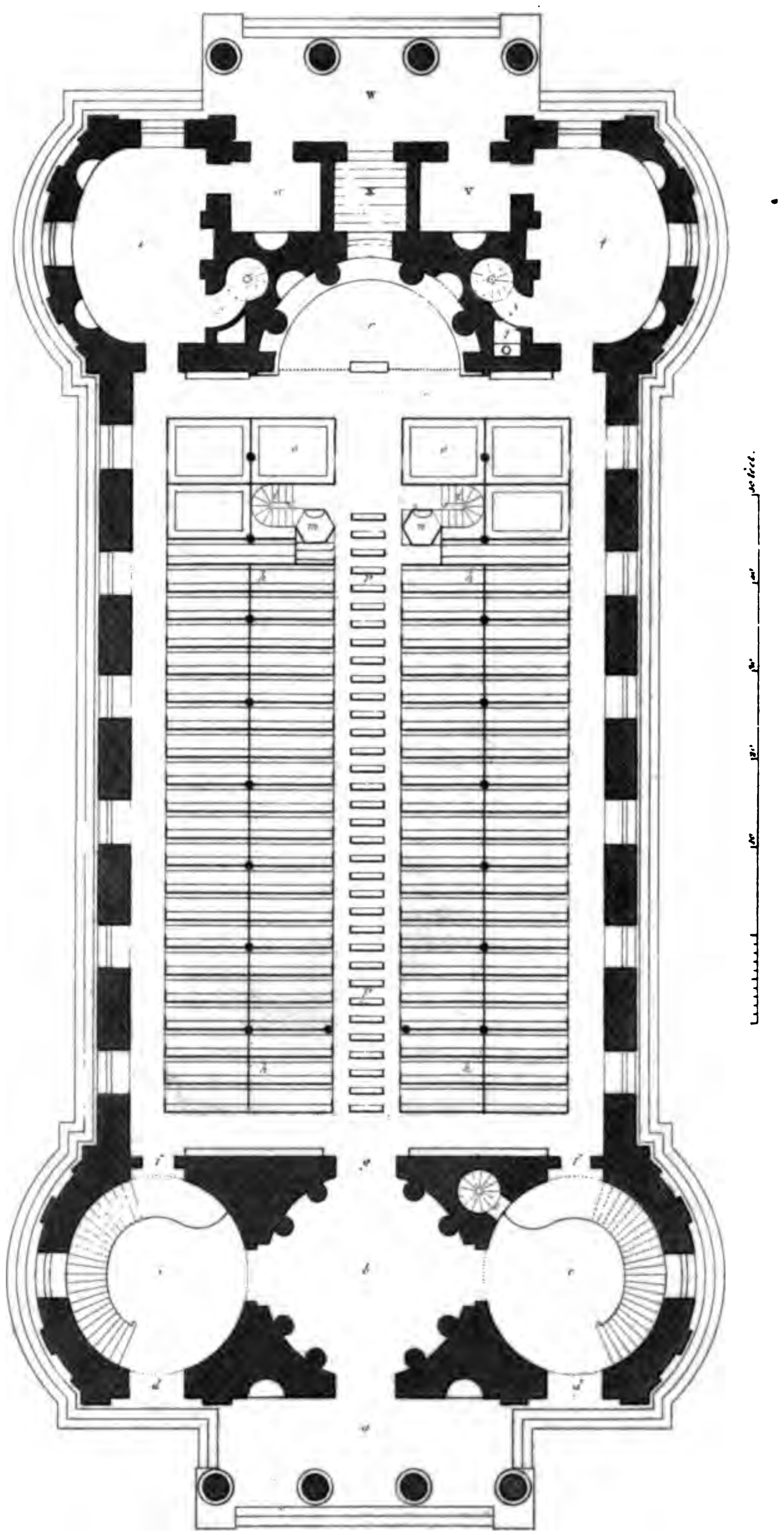
Designed by M. A. Nicholson.

Engraved by E.K.

London, Published by Thos Kelly, 17 Paternoster Row, Octr 25, 1823.

PLATE XVI.

GROUND PLAN, OF A CHURCH IN THE GRECIAN STYLE.



Designed by M. A. Nicholson.

Engraved by E. Kinnim.

London. Published by Thos. Kelly, 17 Paternoster Row. Octr. 25. 1825.

DESIGNS FOR PARTITIONS.

PLATE XXIII.

Fig. 1.

Fig. 2.

P. Nicholson.

Engraved by W. Symns

London, Published by Thos. Kelly, 17, Paternoster Row, Jan.y 1, 1822.

Fig. 5.

Fig. 1.

Fig. 2.

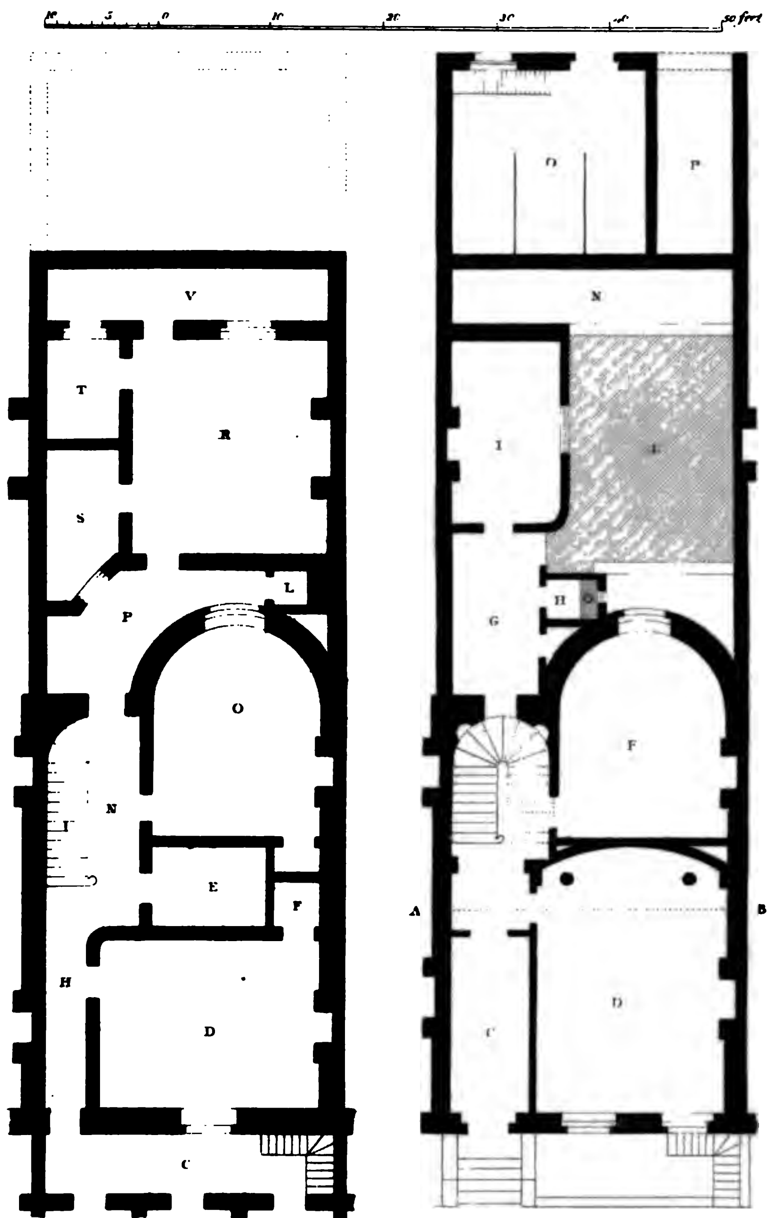
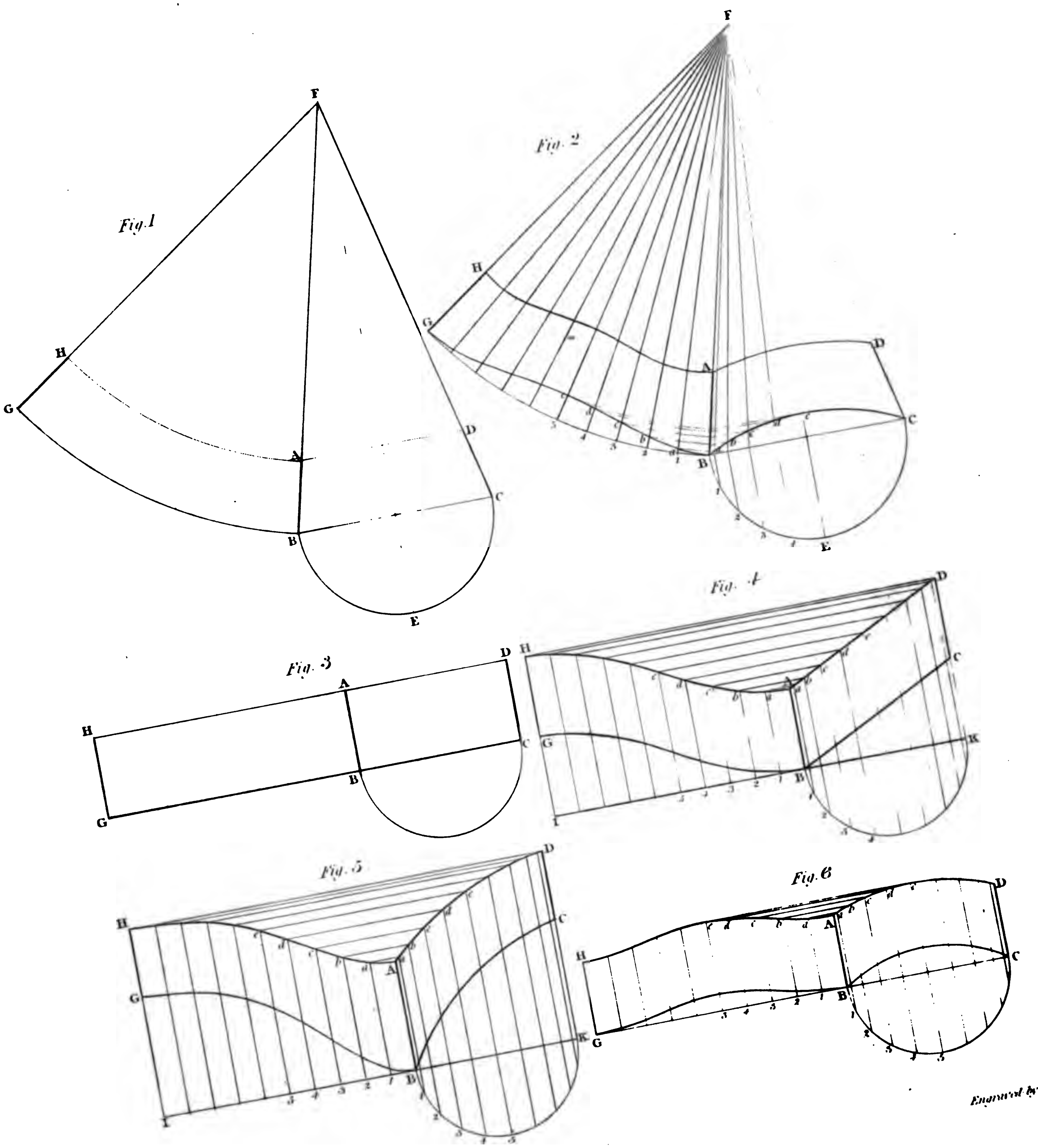


Fig. 3. Fig. 4.

Designed by M.A. Nicholson. Engraved by R. Adlard.

London. Published by Thos. Kelly. 17. Paternoster Row. July 25.1823.

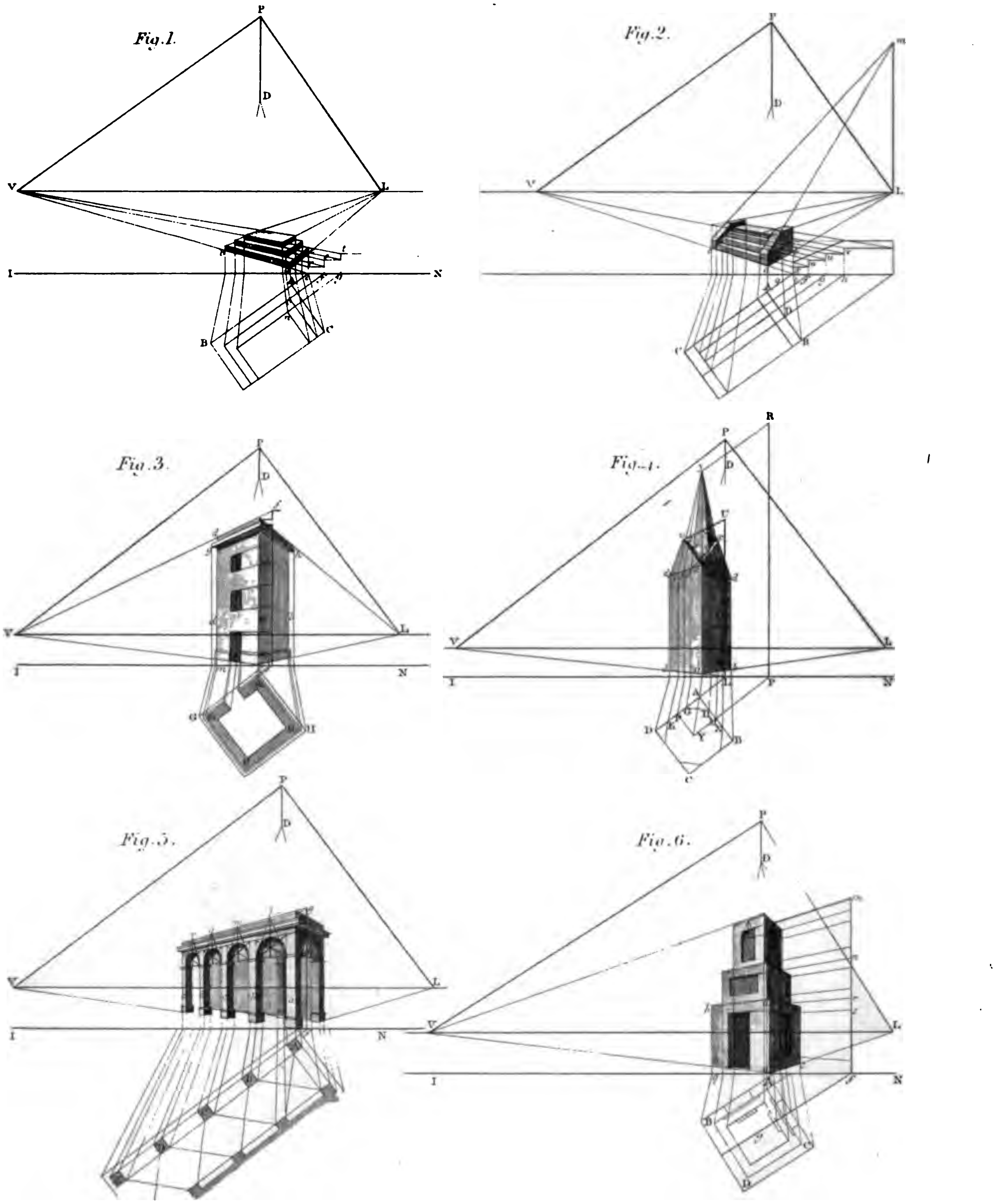
COVERINGS OF SOLIDS.



Engraved by

ublished by Thos Kelly, 17 Paternoster Row, January 1, 18..

PERSPECTIVE.

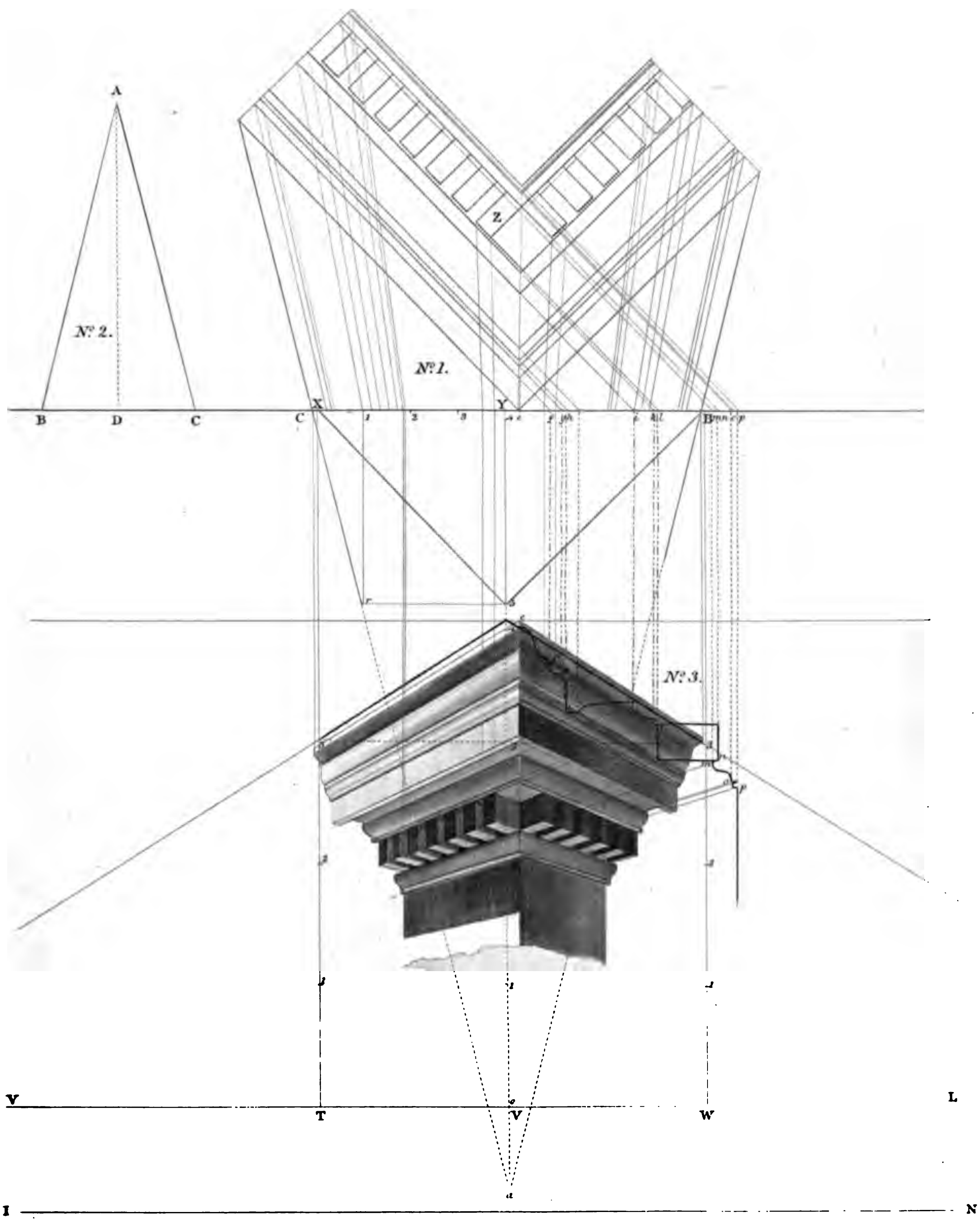


Drawn by M.A.Nicholson.

London, Published by Thos. Kelly 17 Paternoster Row Sep.r 28 1833.

Engraved by W. Symns.

PERSPECTIVE.

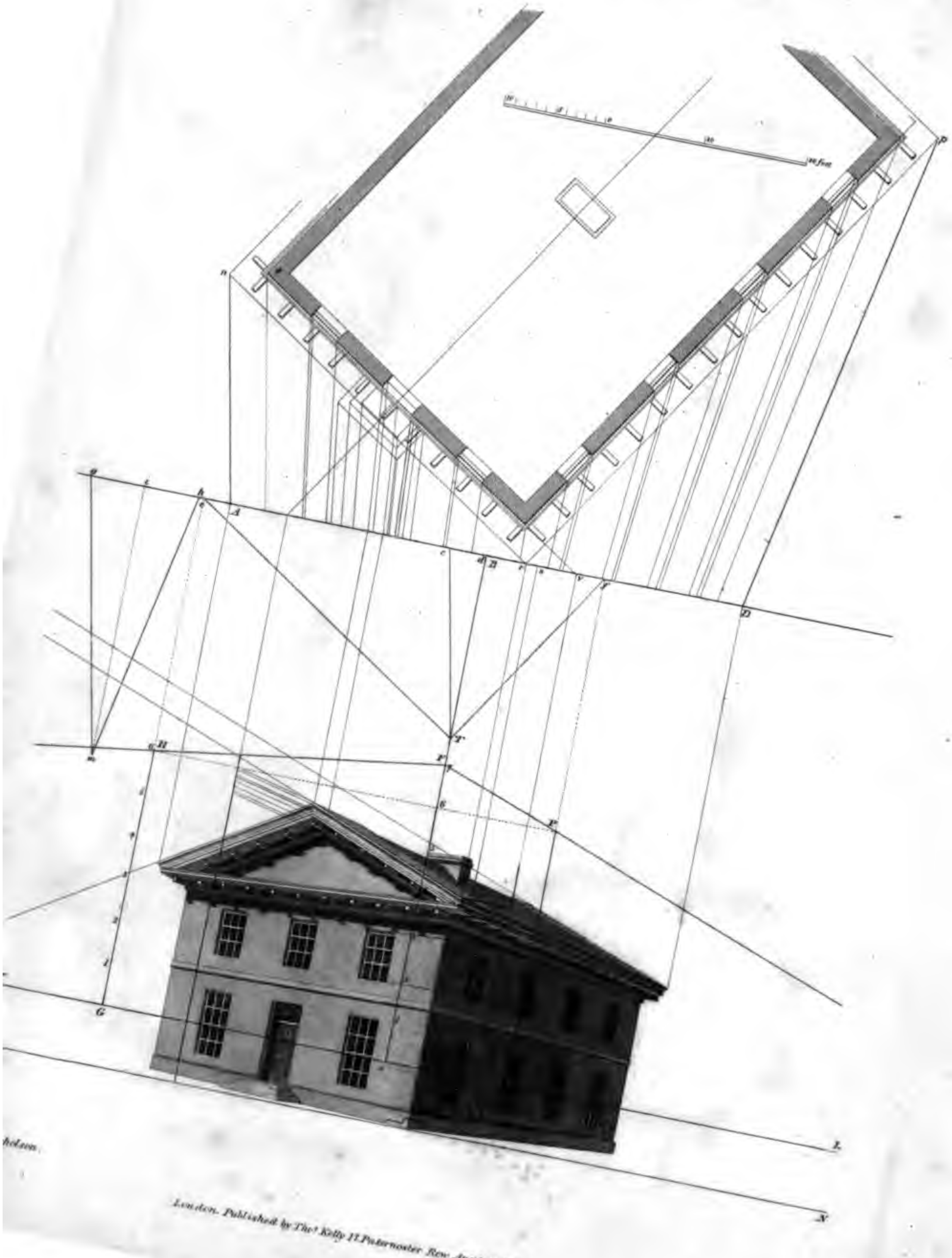


Drawn by M.A. Nicholson.

Engraved by W. Symns

London Published by Thos. Kelly, 17 Paternoster Row. Sept. 13. 1823.

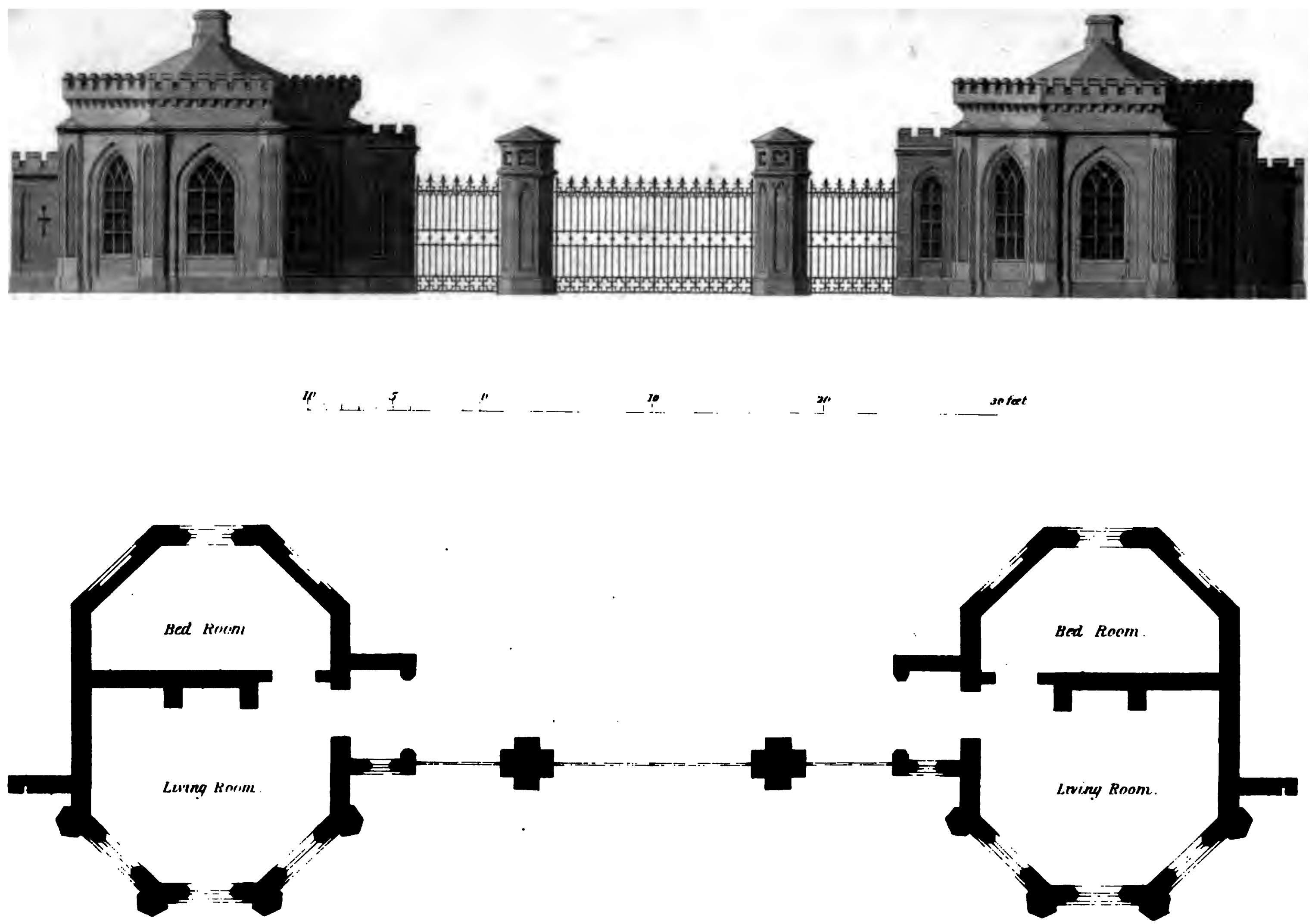
PERSPECTIVE.



…holson.

London. Published by Thos. Kelly 17 Paternoster Row Aug.t 22 1823.

Engraved by H. Adlard.

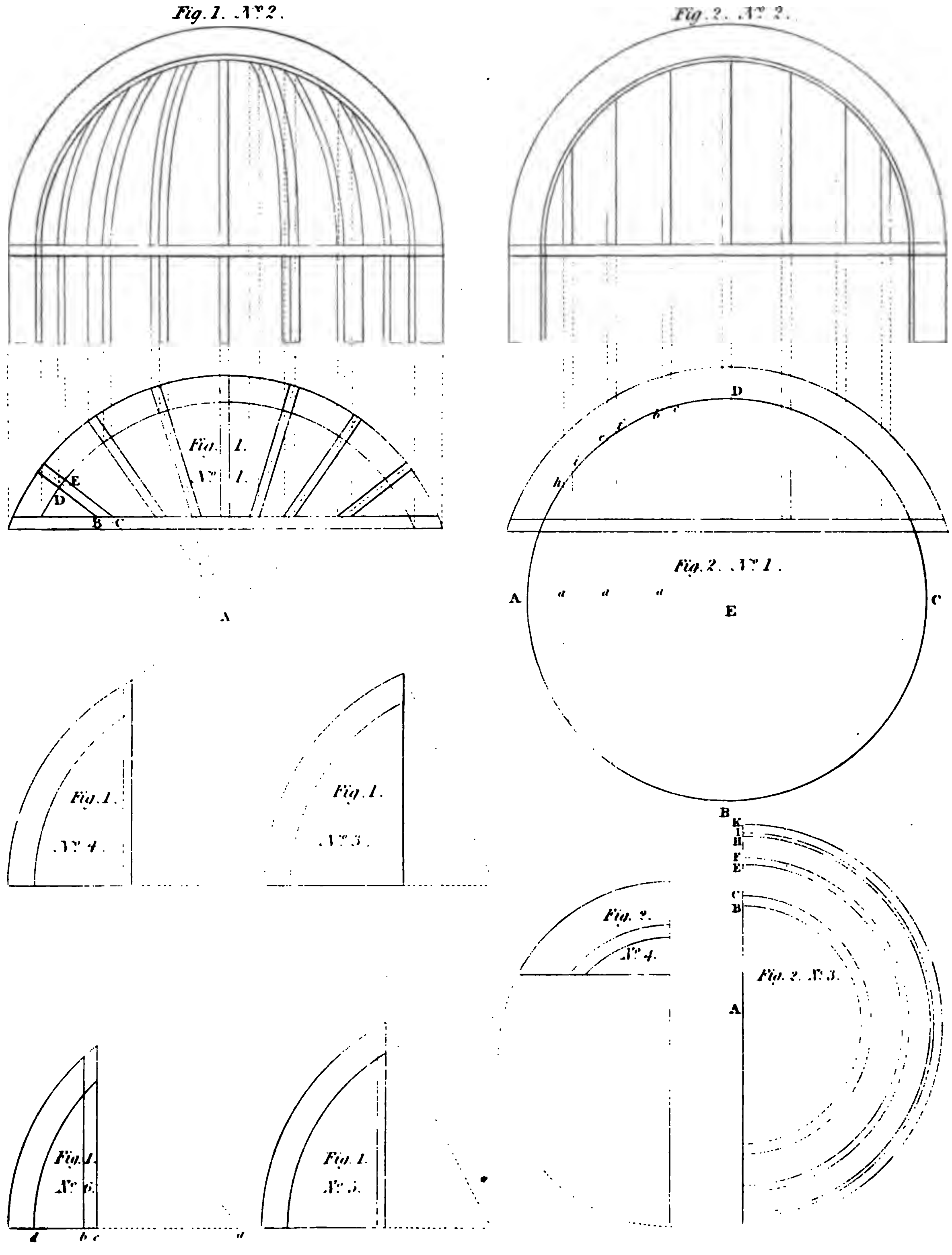


Designed by M.A Nicholson.

Engraved by H. Adlard.

London. Published by Thomas Kelly, 17, Paternoster Row. August 16, 1823.

NICHES.

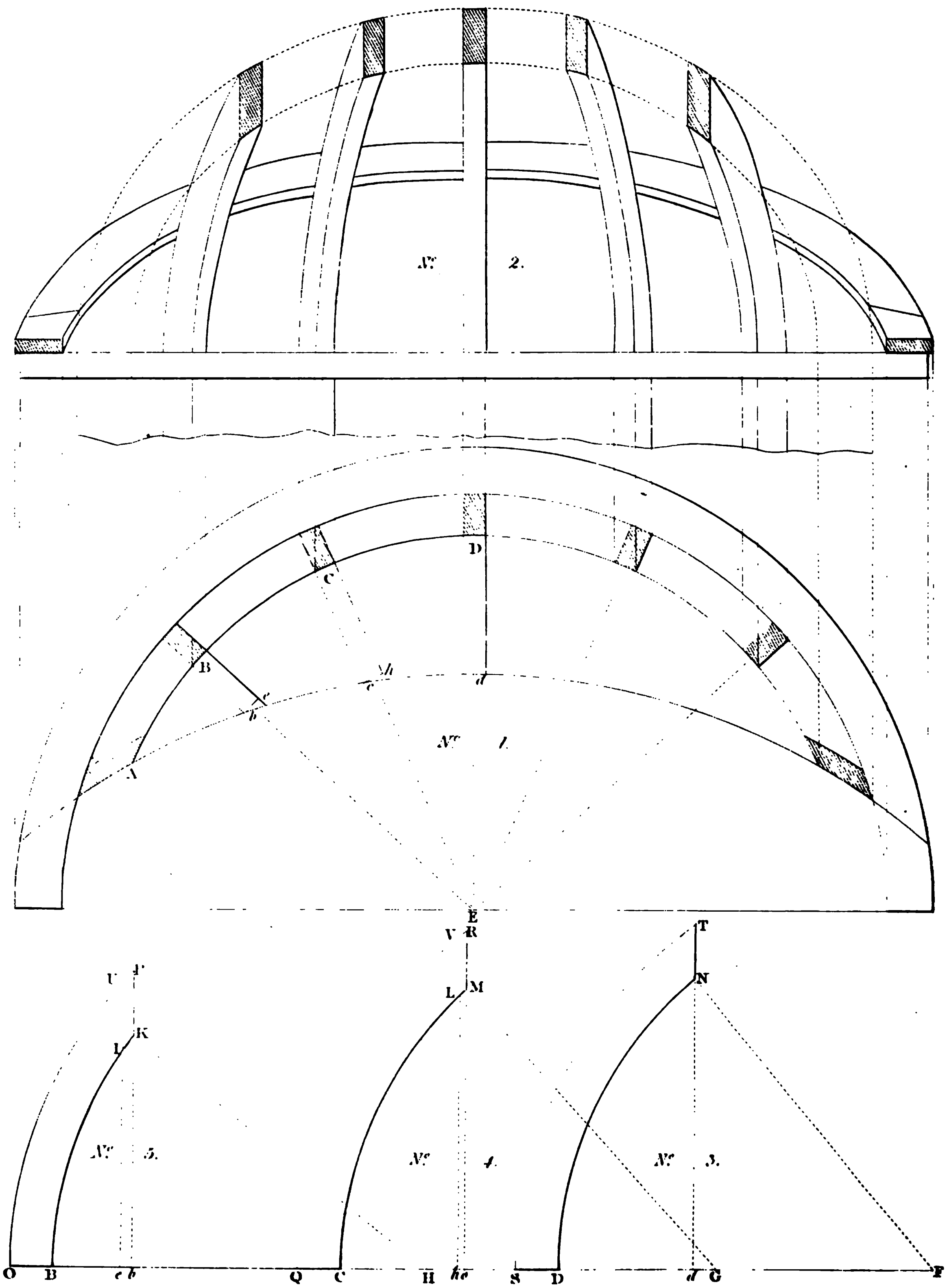


Drawn by P. Nicholson.

Engraved by W. Symns.

London. Published by Thos. Kelly, 17 Paternoster Row, January 1 [illegible].

NICHES.



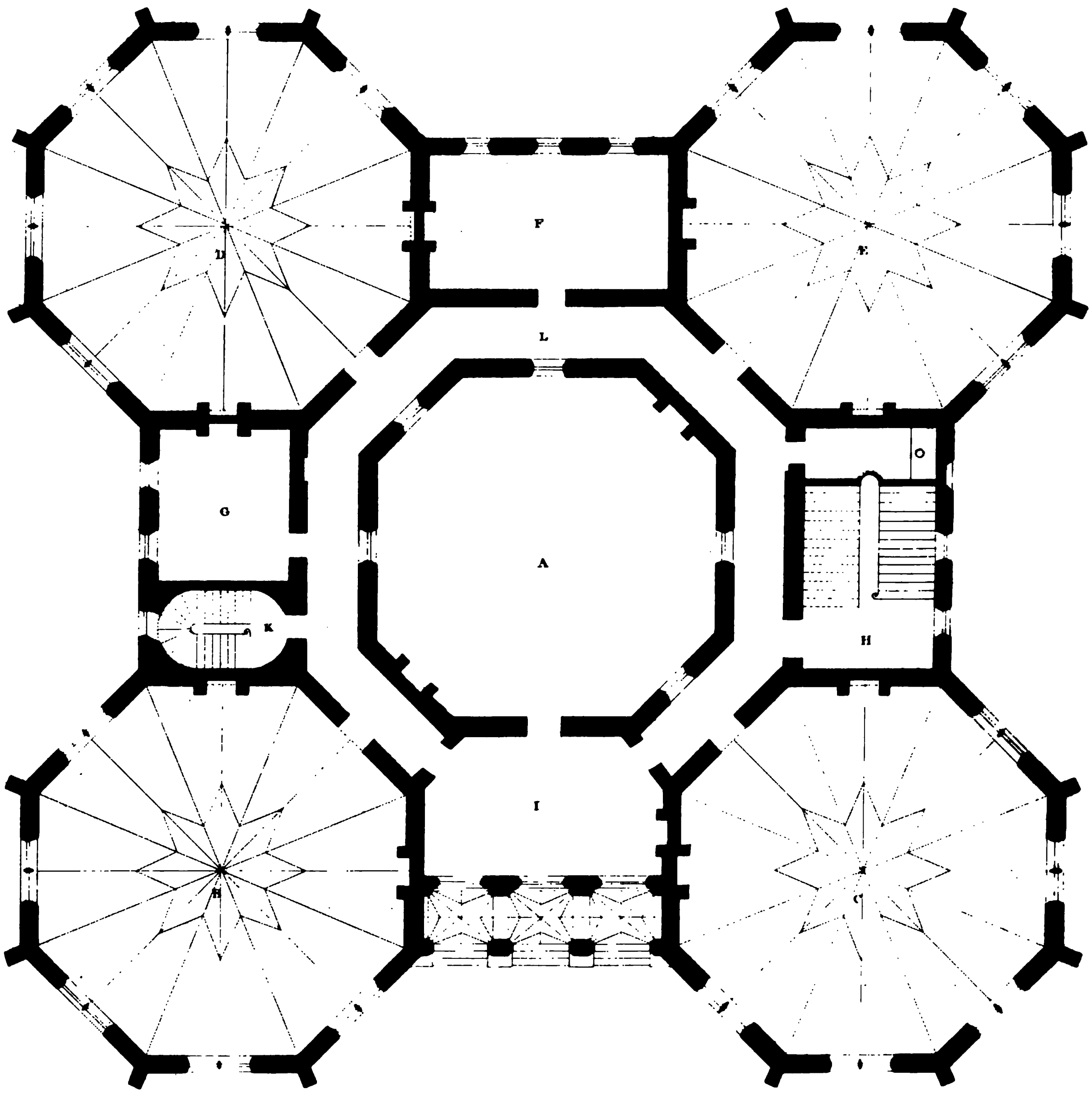
Drawn by M.A. Nicholson.

Engraved by W. Symns.

London, Published by Thos. Kelly, 17 Paternoster Row, January 1, 1823.

GROUND PLAN OF A DESIGN FOR A MANSION.

IN THE CASTELLATED STYLE.

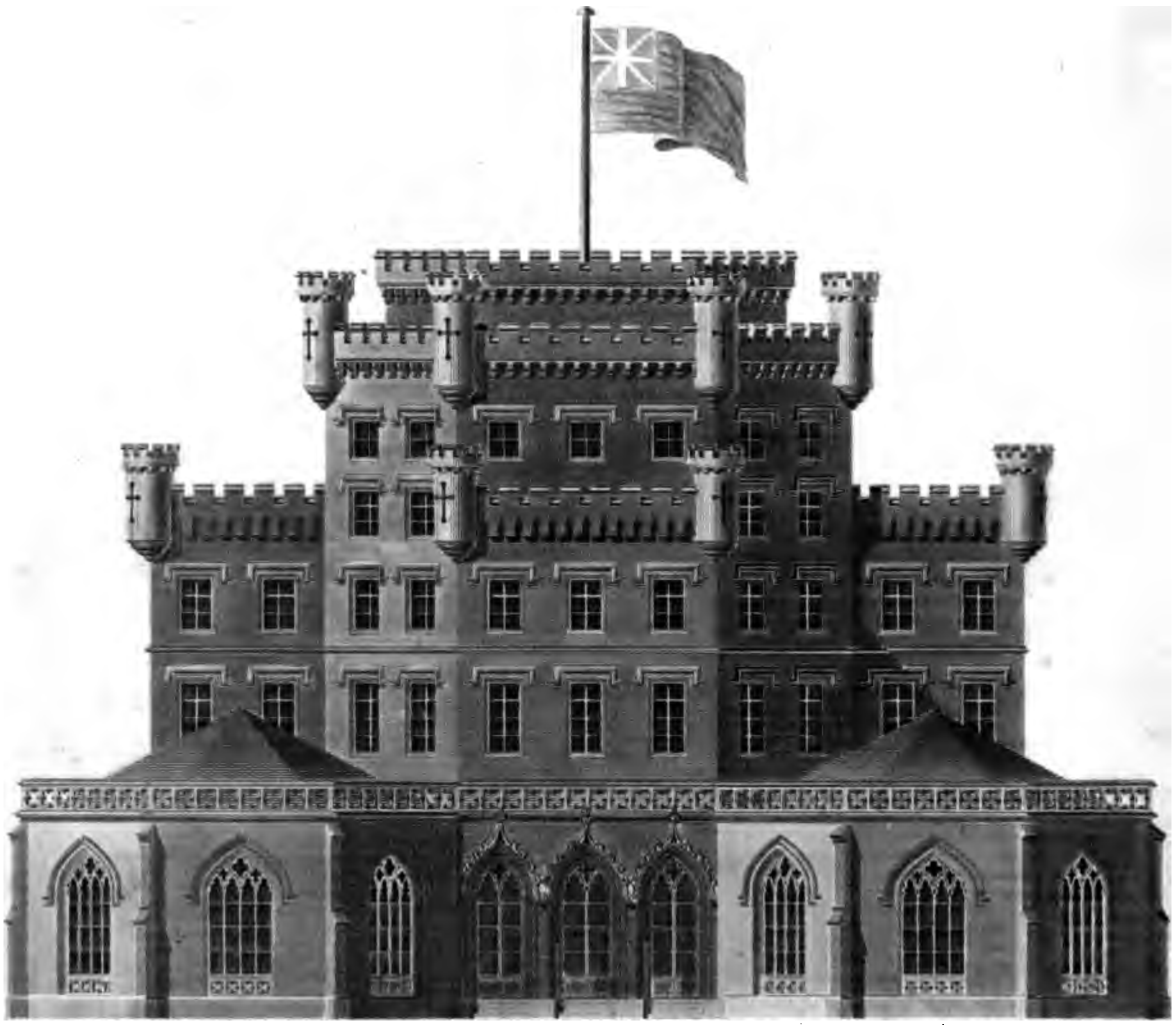


signed by M.A.Nicholson.

Engraved by R.Roffe.

London Published by Tho. Kelly, 17 Paternoster Row June 1 1823.

DESIGN FOR A MANSION IN THE CASTELLATED STYLE.



Designed by M. A. [illegible] *Engraved by [illegible]*

London, Published by Thos. Kelly, 17 Paternoster Row Sept. 20, 1823.

ELEVATION.

Fig. 1.

CHAMBER PLAN

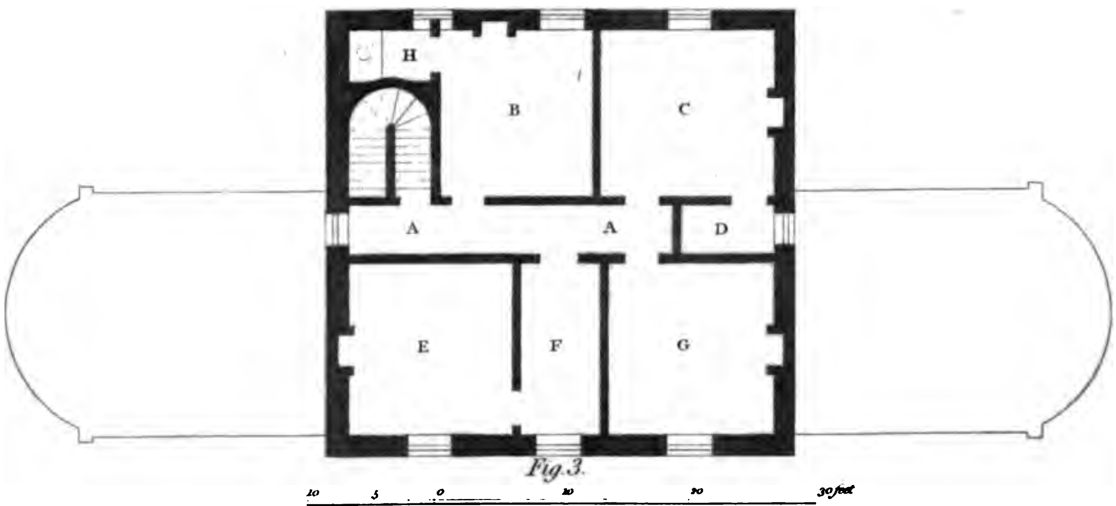


Fig. 3.

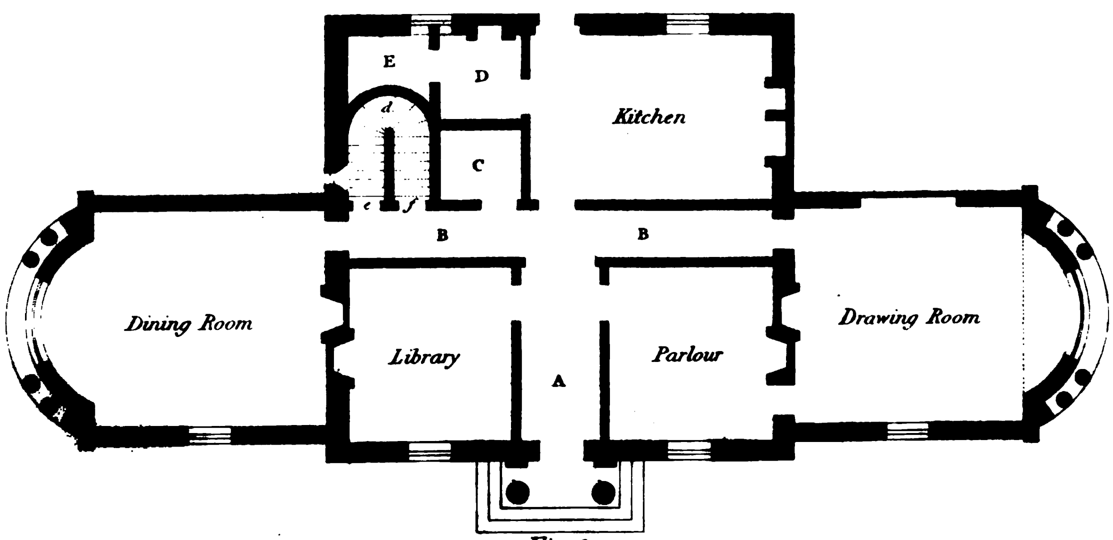


Fig. 2.

GROUND PLAN

Designed by M.A. Nicholson. Engraved by W. Symns.

London, Published by Thos. Kelly, 17, Paternoster Row, Augst. 9, 1823.

. ELEVATION .

Fig. 1.

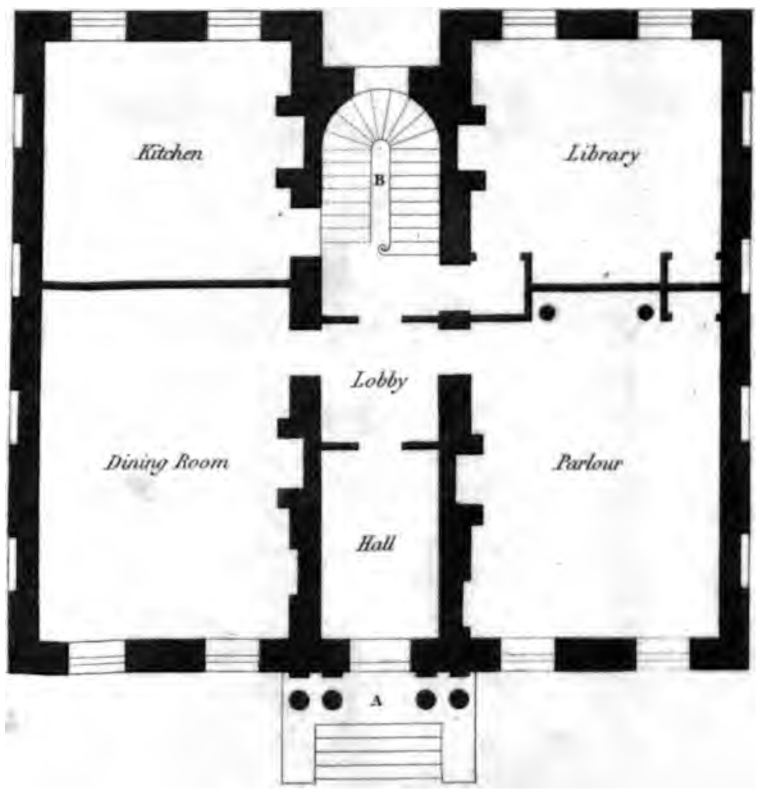


Fig. 2.

GROUND PLAN .

n by M.A. Nicholson. Engrava

London. Published by Thos. Kelly. 17 Paternoster Row June 1 1823.

ELEVATION.

Fig. 1.

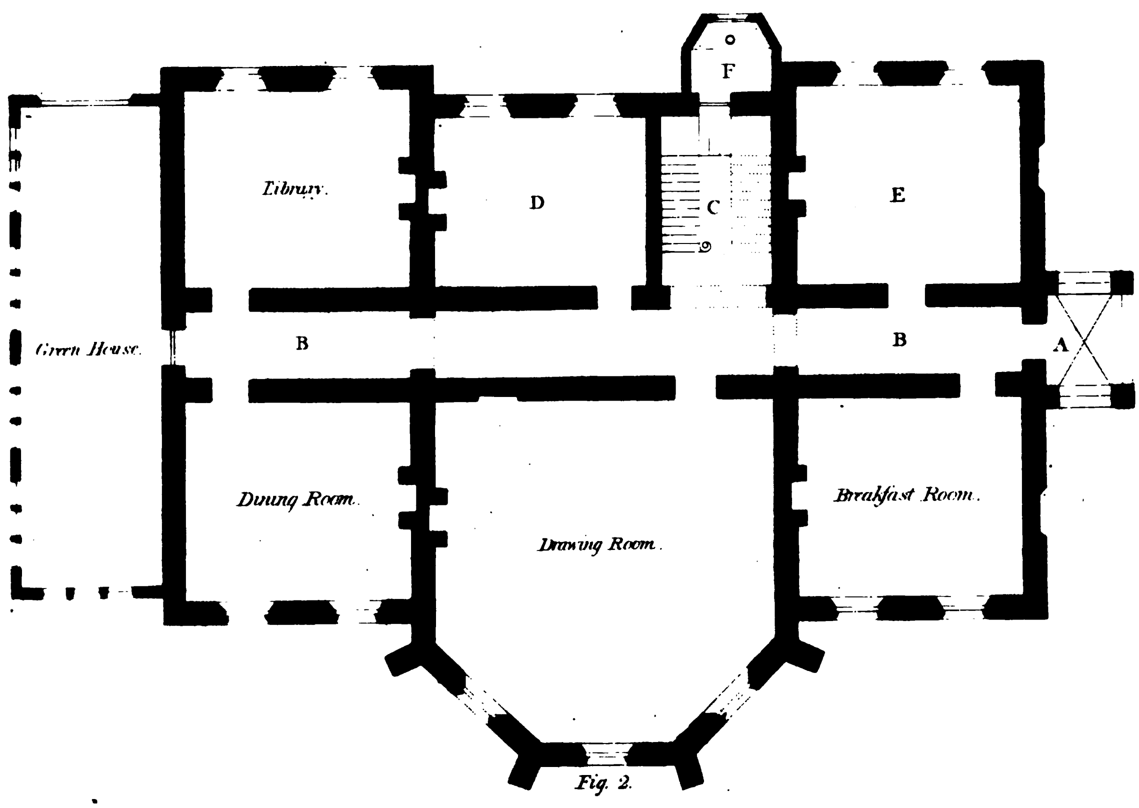


Fig. 2.

GROUND PLAN

Designed by M.A. Nicholson.

Engraved by W. Symns.

London. Published by Thos. Kelly. 17 Paternoster Row. July 25. 1823.

ELEVATION,

Fig. 1.

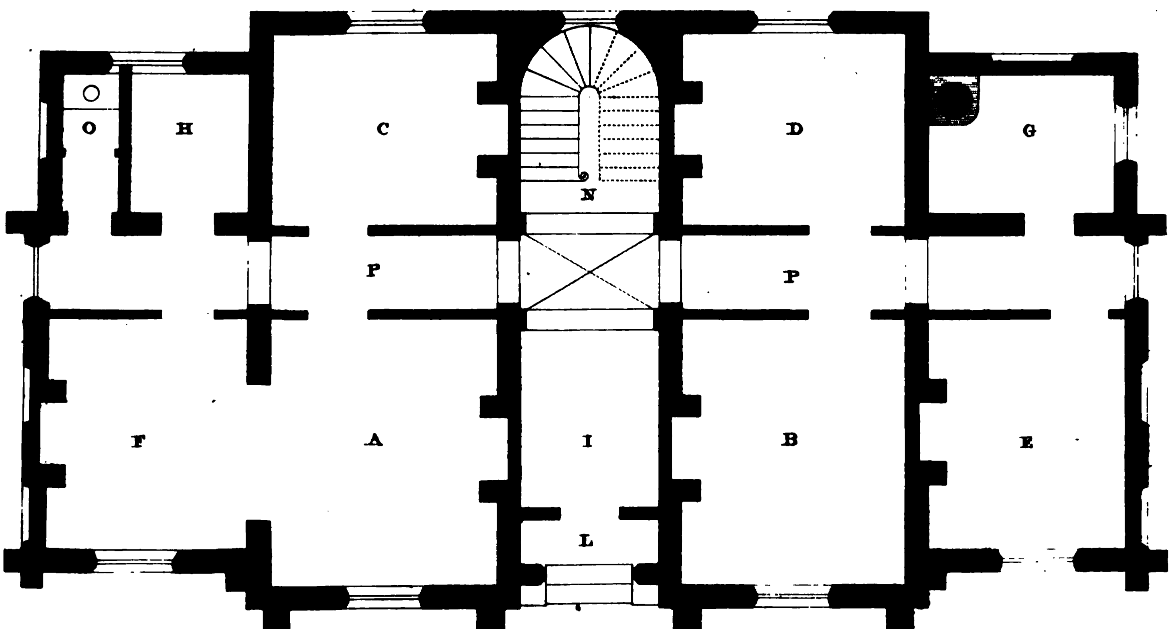


Fig. 2.

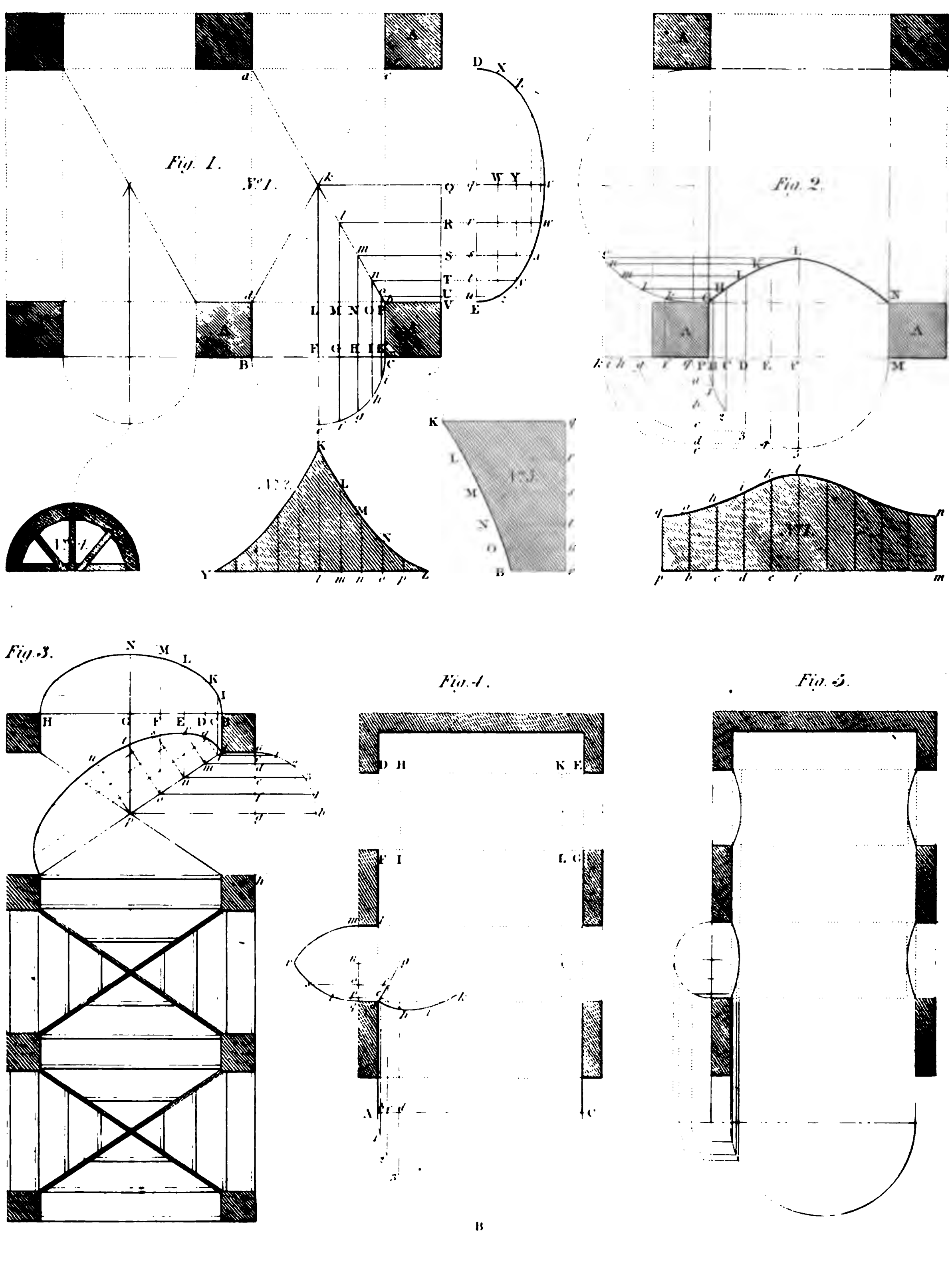
GROUND PLAN,

A. Nicholson.

Engraved by J

London, Published by Thos. Kelly, 17 Paternoster Row, August 1, 1823.

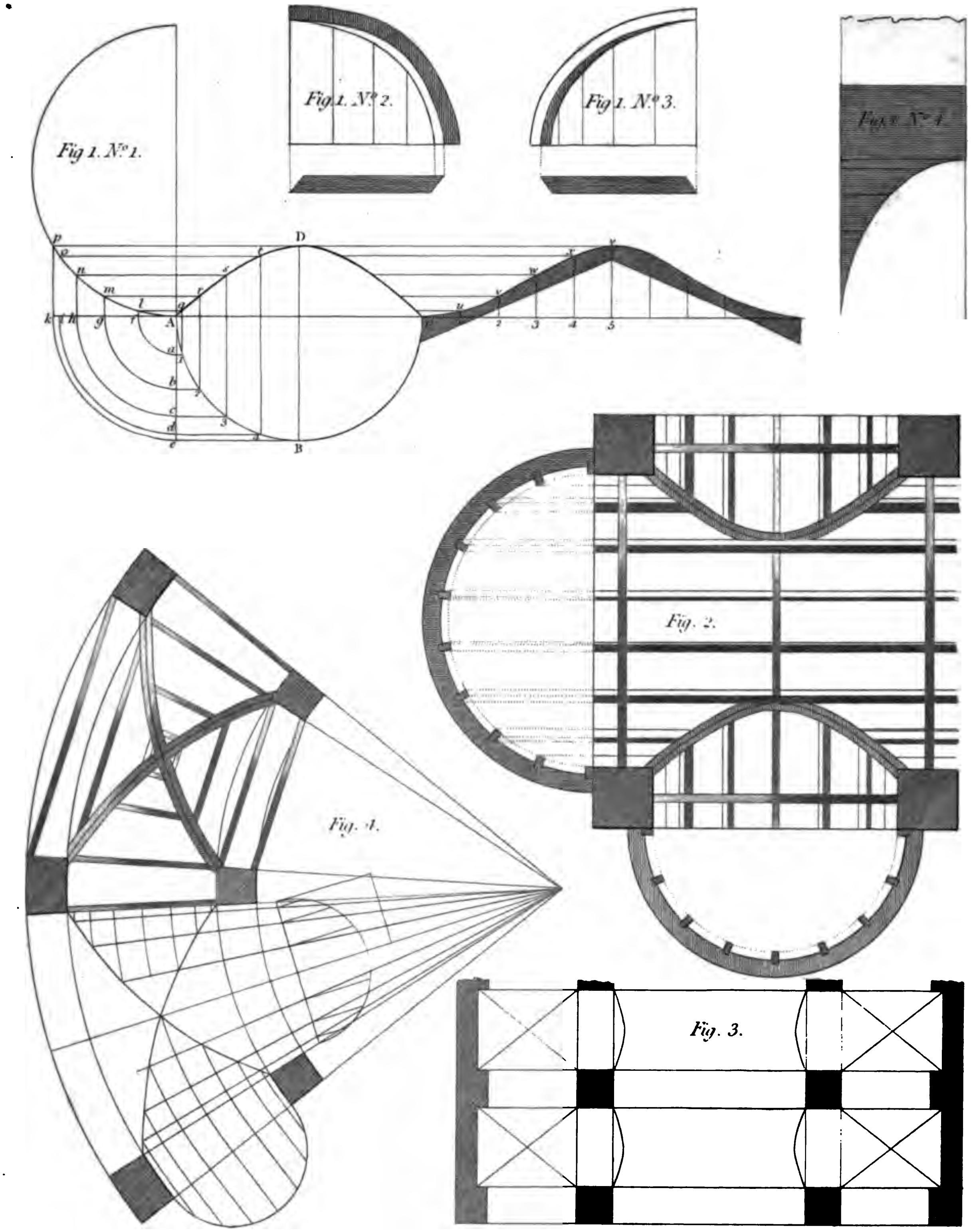
GROINS AND ARCHES.



Drawn by P. Nicholson.

London, Published by Thos. Kelly, 17 Paternoster Row, January 1, 1822.

Engraved by W. Symns

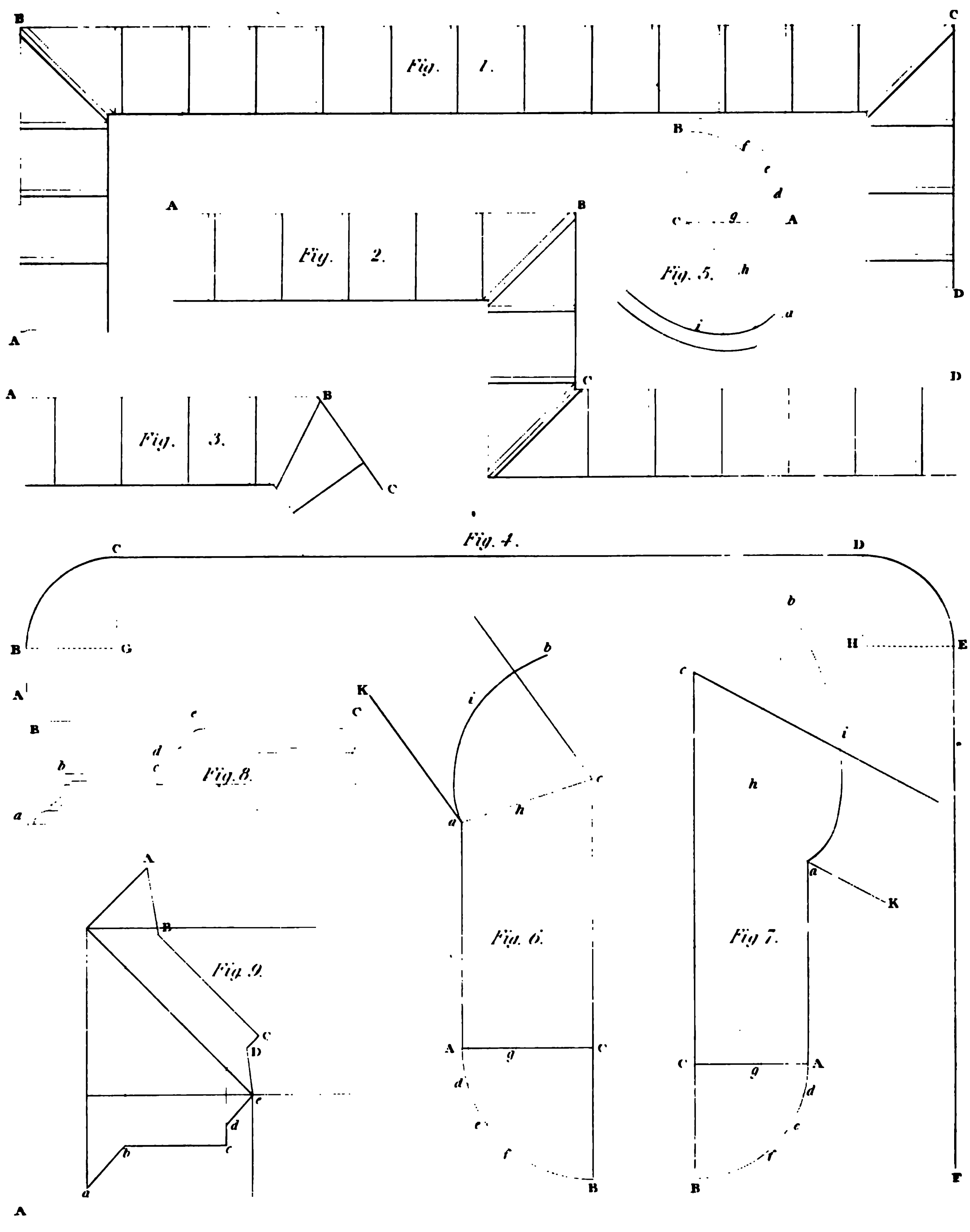


Drawn by P. Nicholson.

Engraved by W. Symns.

London, Published by Thos. Kelly, 17, Paternoster Row, Jany. 1, 1822.

BRACKETING FOR CORNICES AND COVES.

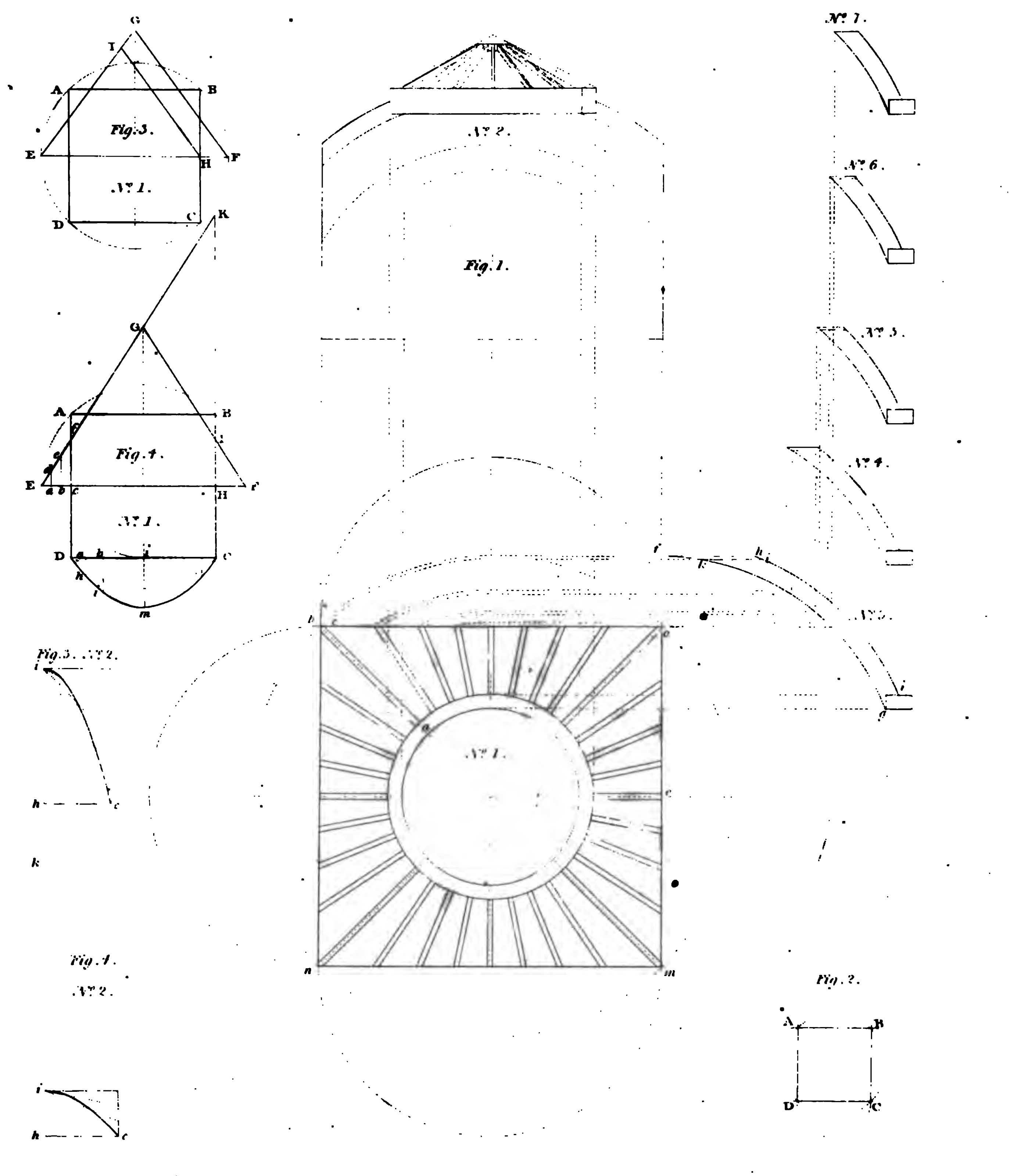


Drawn by P. Nicholson

Engraved by W. Symns

London, Published by Thos Kelly, 17 Paternoster Row, January 18[illegible]

PENDENTIVE BRACKETING.



rawn by P. Nicholson.

Engraved by W. Jyrn

London, Published by Thos Kelly 17 Paternoster Row January 1. 1822.

DOMES.

Fig. 1. No. 1.

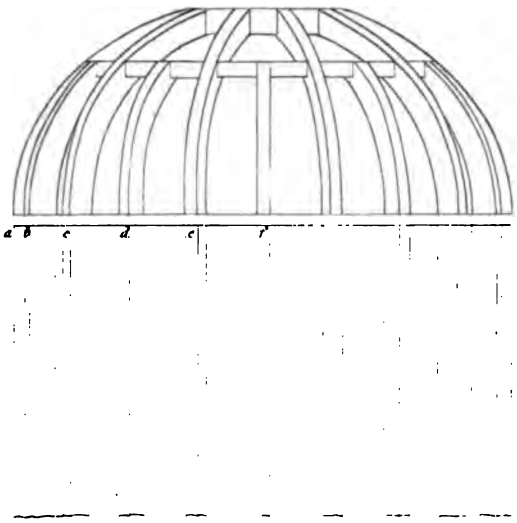


Fig. 2. No. 1.

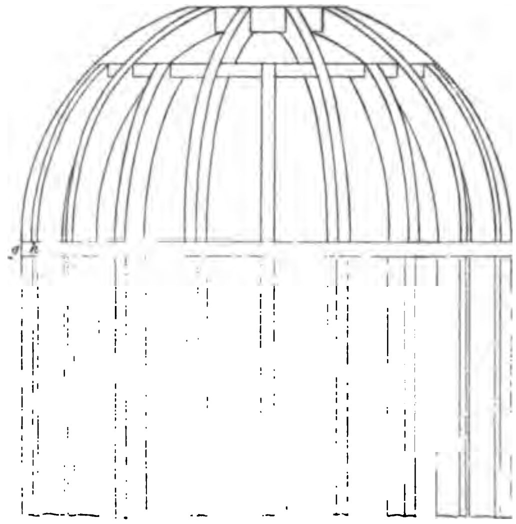


Fig. 1. No. 2.

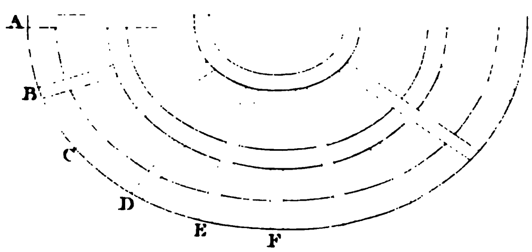
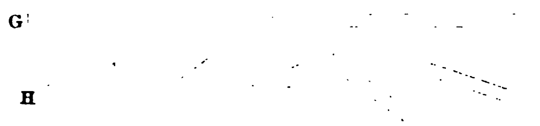


Fig. 2. No. 2.



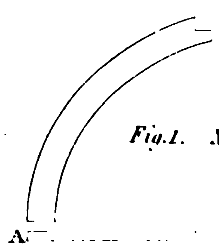


Fig. 1. No. 3.

A

Fig. 1. No. 4.

B

Fig. 2. No. 3.

G

Fig. 2. No. 4.

H

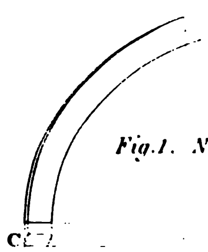


Fig. 1. No. 5.

C

Fig. 1. No. 6.

D

Fig. 1. No. 7.

E

Fig. 1. No. 8.

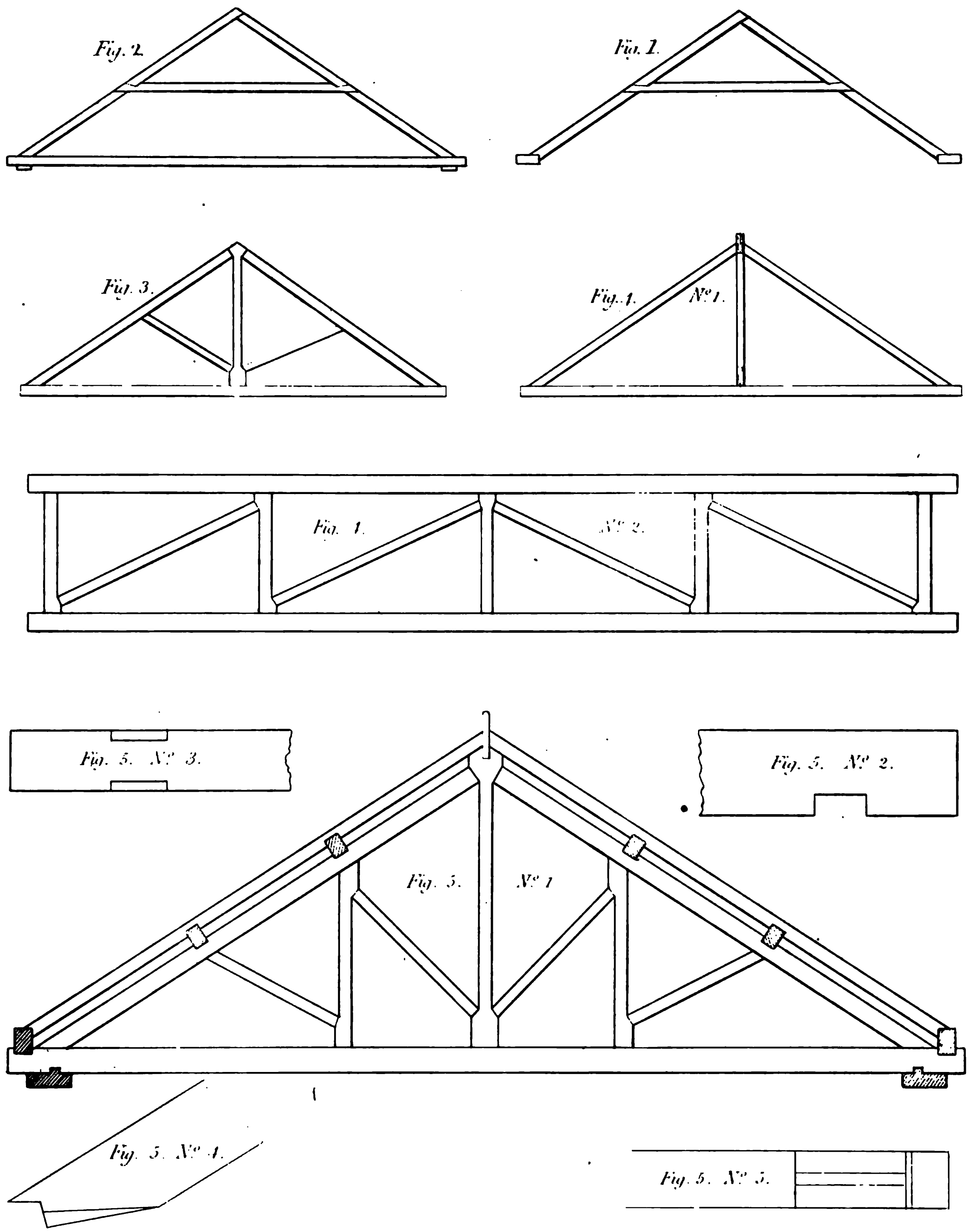
F

Drawn by M. A. Nicholson.

Engraved by W. Symns.

London. Published by Thos. Kelly 17 Paternoster Row November 1. 1822.

DESIGNS FOR ROOFS.

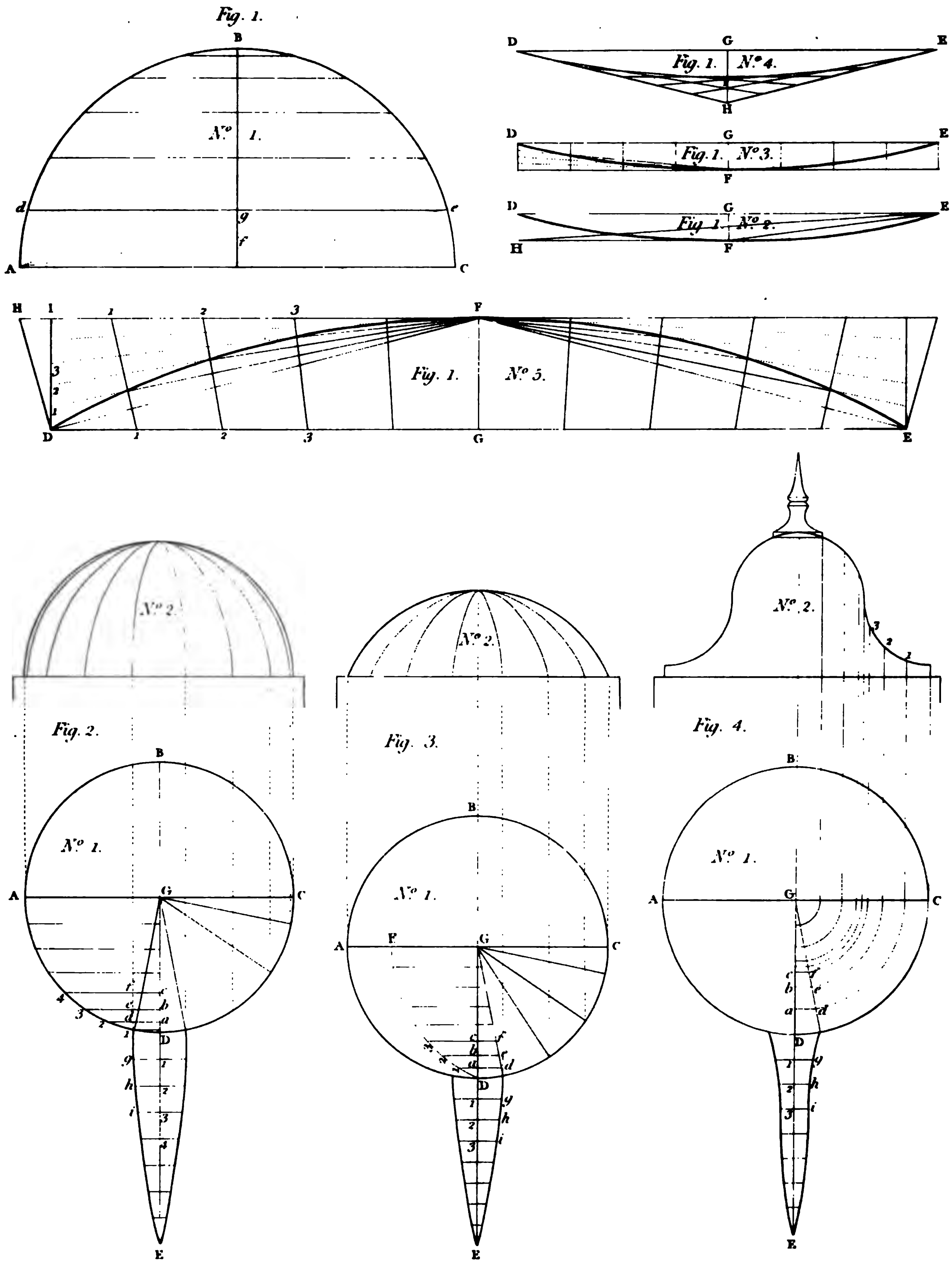


Drawn by P. Nicholson. Engraved by W. Symns.

London, Published by Thos. Kelly, 17, Paternoster Row, Jany. 1, 1822.

PLATE XVI.

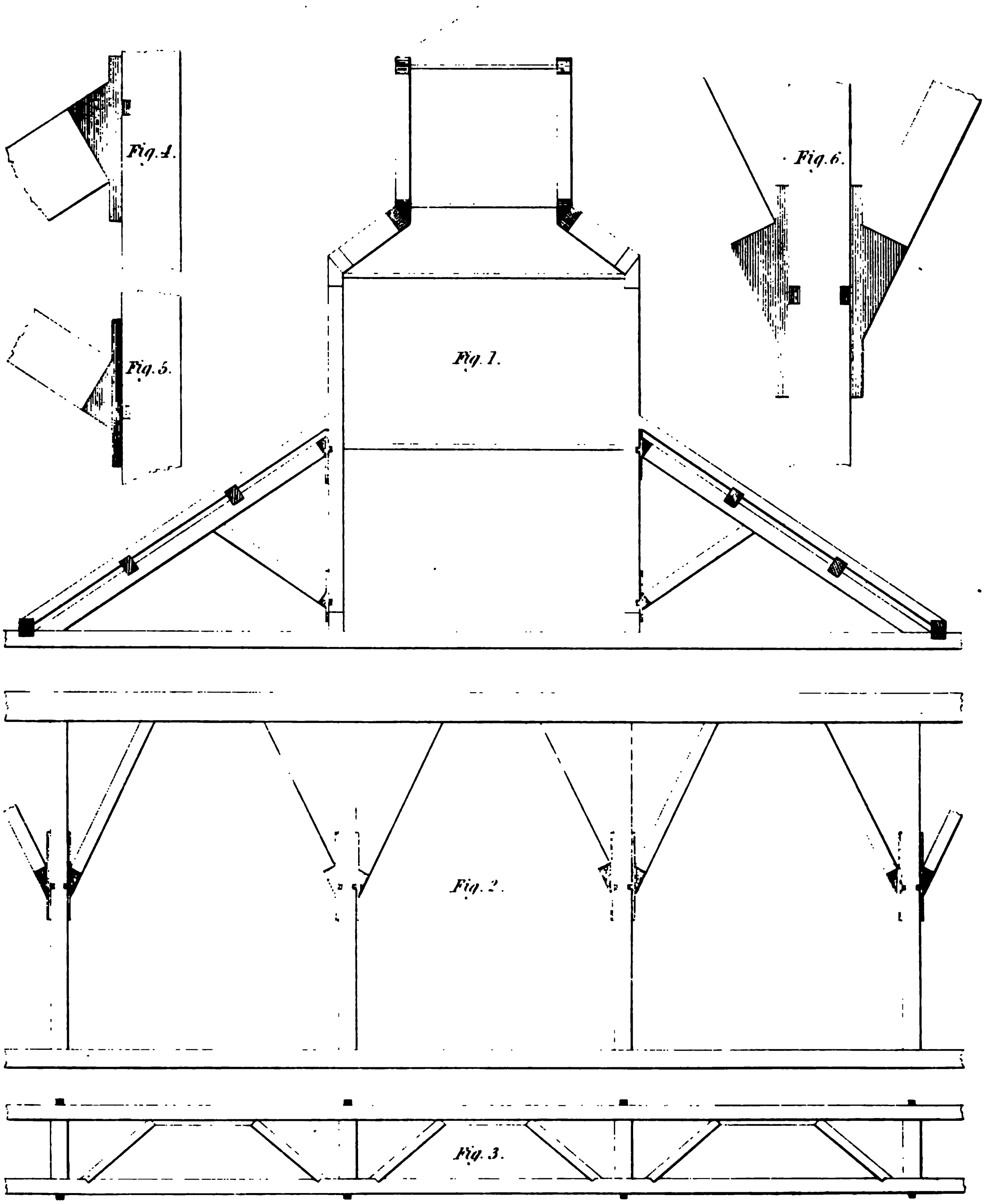
COVERINGS OF CIRCULAR ROOFS.



Drawn by P. Nicholson.

Engraved by W. Symns.

London, Published by Thos. Kelly, 17, Paternoster Row, Jan.y 1, 1822.



Drawn by P. Nicholson.

Engraved by W. Symns.

London, Published by Thos. Kelly, 17 Paternoster Row, January 1, 1822.

Designs for Shutting Windows.

Fig. 1.

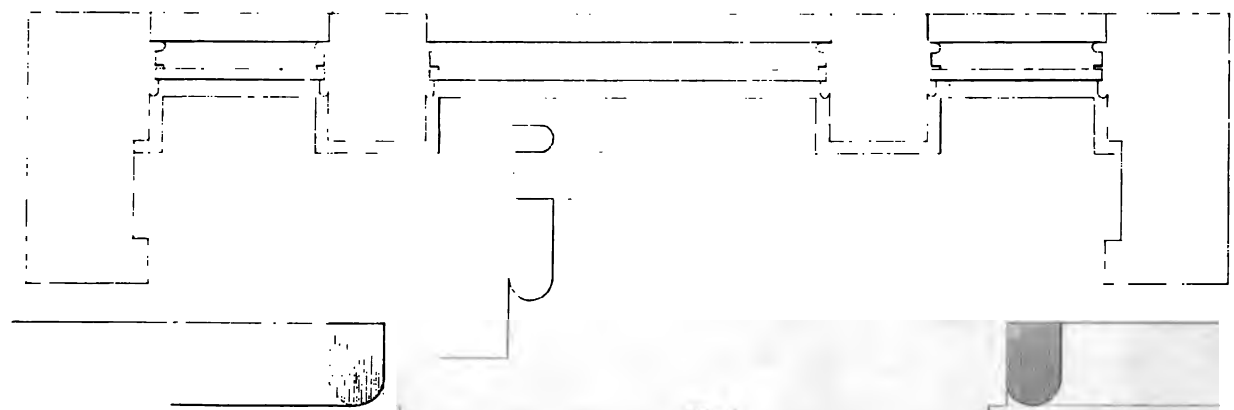


Fig. 2.

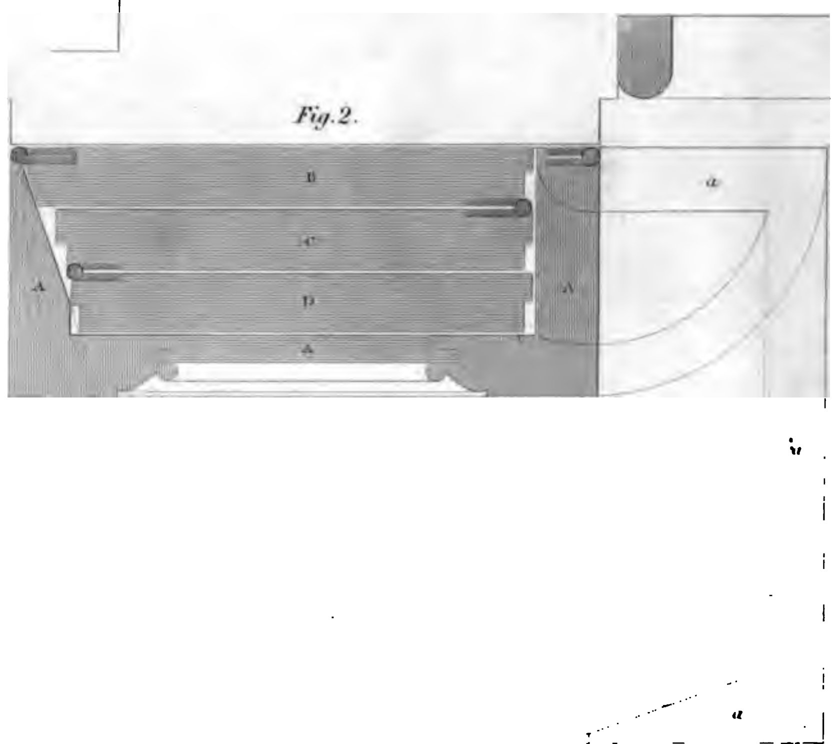


Fig. 4.

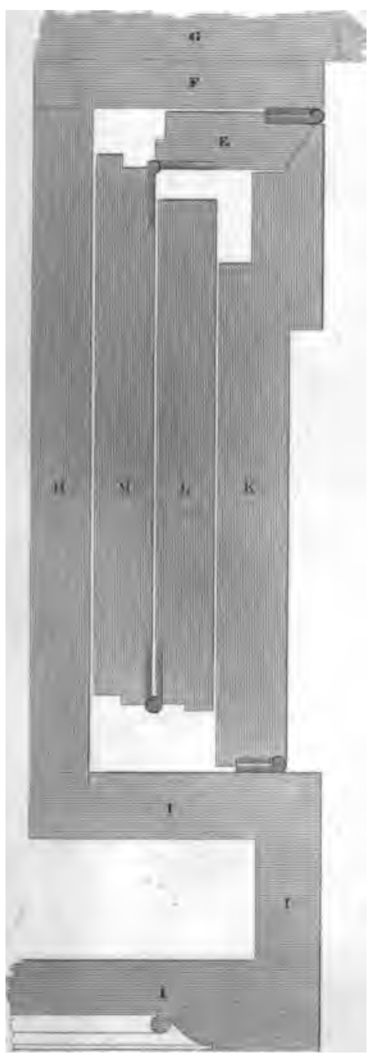
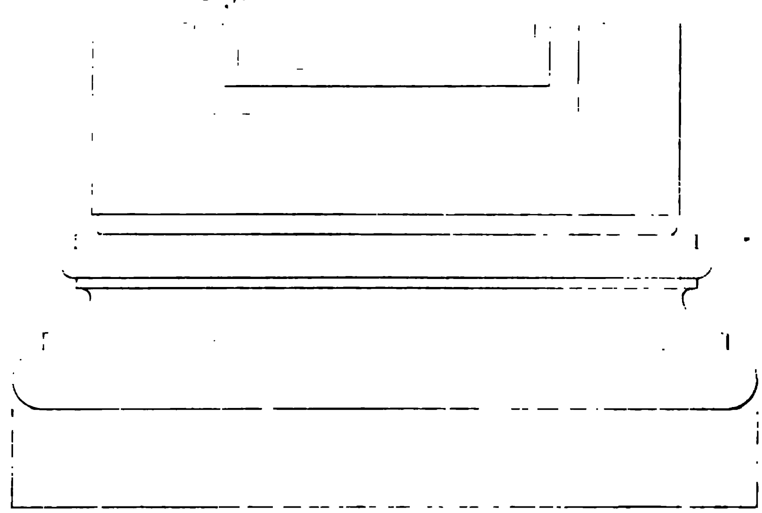


Fig. 3.



awn by P. Nicholson.

London, Published by Thos Kelly 17 Paternoster Row Sep.t 16 1823

Engraved b

STAIRS.

Fig. 1.

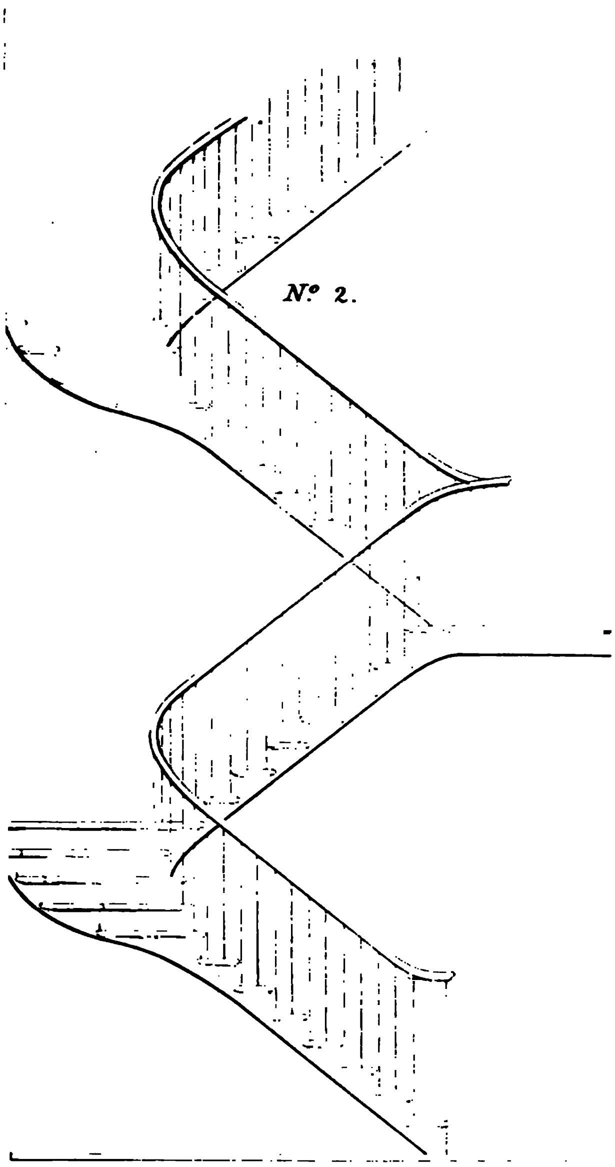
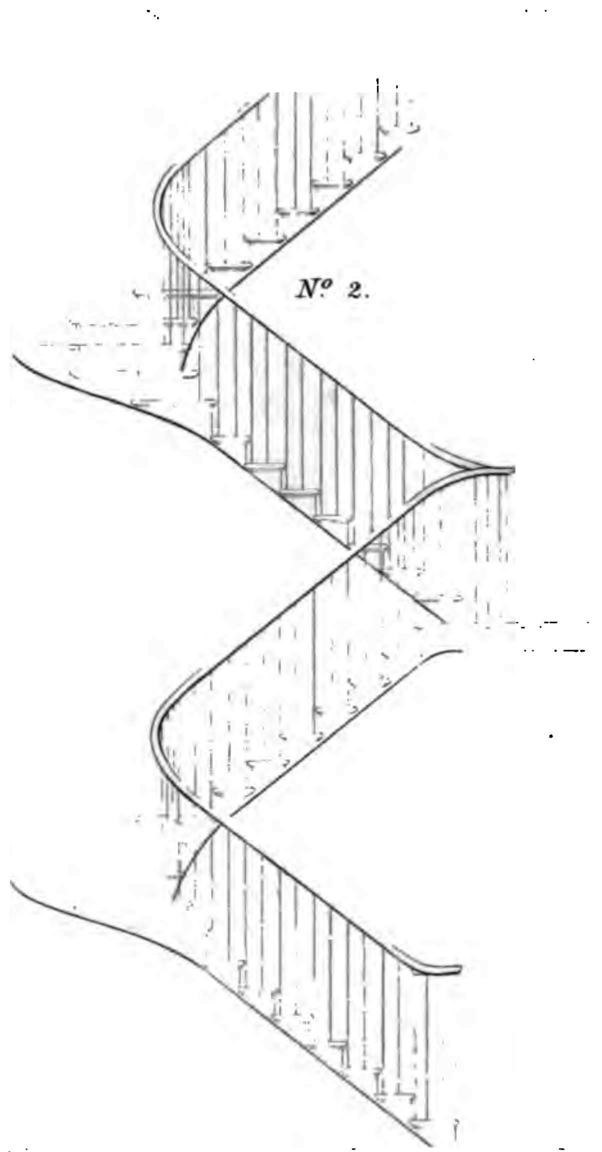
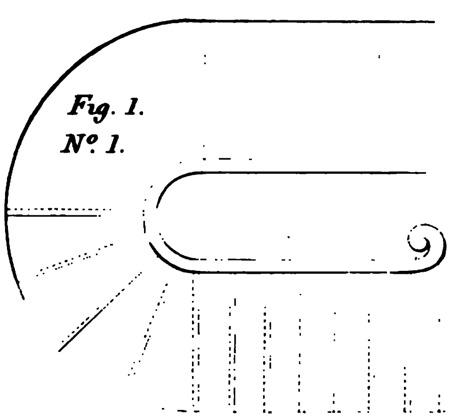
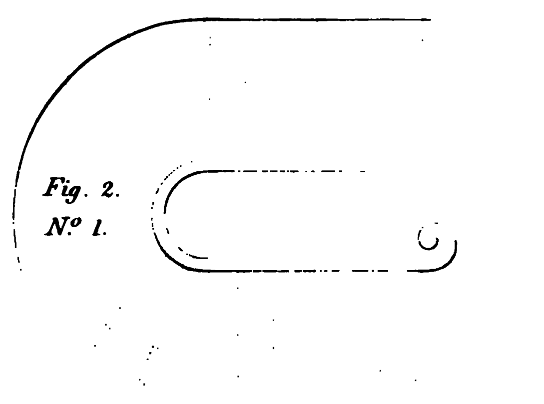


Fig. 2.







Drawn by P. Nicholson.

London. Published by Tho.s Kelly, 17 Paternoster Row January 1. 1822.

Engraved by W. Symns